

# A Deformation of Sasakian Structure in the Presence of Torsion and Supergravity Solutions

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## Abstract

We discuss a deformation of Sasakian structure in the presence of totally skew-symmetric torsion by introducing odd dimensional manifolds whose metric cones are Kähler with torsion. It is shown that such a geometry inherits similar properties to those of Sasakian geometry. As an example of them, we present an explicit expression of local metrics and see how Sasakian structure is deformed by the presence of torsion. We also demonstrate that our example of the metrics admits the existence of hidden symmetries described by non-trivial odd-rank generalized closed conformal Killing-Yano tensors. Furthermore, using these metrics as an *ansatz*, we construct exact solutions in five dimensional minimal (un-)gauged supergravity and eleven dimensional supergravity. Finally, we discuss the global structures of the solutions and obtain regular metrics on compact manifolds in five dimensions, which give natural generalizations of Sasaki–Einstein manifolds  $Y^{p,q}$  and  $L^{a,b,c}$ . We also discuss regular metrics on non-compact manifolds in eleven dimensions.

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## I. INTRODUCTION

Sasakian geometry [1] has attracted intense interest in theoretical and mathematical physics since the application was found in higher-dimensional supergravity theories, string theories and M-theory. Arguably, the most important examples are Sasaki–Einstein manifolds which have been discussed in the context of the AdS/CFT correspondence, especially in the physically interesting dimensions five and seven. In five dimensions, the most simplest Sasaki–Einstein manifold is the standard round 5-sphere, denoted by  $S^5$ , and it provides a supersymmetric background  $\text{AdS}_5 \times S^5$  of type IIB supergravity theory, on which D3-brane physics is conjectured to be dual of an  $\mathcal{N} = 4$  four dimensional superconformal field theory [2]. More general five dimensional Sasaki–Einstein manifolds  $M_5$  provide a variety of supersymmetric backgrounds  $\text{AdS}_5 \times M_5$ , which are in general dual of  $\mathcal{N} = 1$  superconformal field theories. Recently, it was proposed by [3] that  $\mathcal{N} = 6$  three dimensional Chern–Simons–matter theory is related to M2-brane physics on a background  $\text{AdS}_4 \times S^7/\mathbb{Z}_k$  of M-theory. This motivates us to extend  $S^7$  to general seven dimensional Sasaki–Einstein manifolds  $M_7$  and study the backgrounds  $\text{AdS}_4 \times M_7$  corresponding to  $\mathcal{N} = 2$  Chern–Simons theories. Thanks to such proposals, we have now a number of concrete examples of Sasaki–Einstein manifolds. Until recent years, the only explicit examples of Sasaki–Einstein manifolds were  $S^5$  and  $T^{1,1}$  in five dimensions and  $M^{3,2}$ ,  $Q^{1,1,1}$  and  $V^{5,2}$  in seven dimensions. Gauntlett, Martelli, Sparks and Waldram constructed the infinite families of inhomogeneous Sasaki–Einstein manifolds  $Y^{p,q}$  in five [4] and higher [5] odd dimensions. Further generalizations were constructed in all odd dimensions [6–8].

In the familiar story in type IIB supergravity theory,  $\text{AdS}_5 \times M_5$  backgrounds are given as supersymmetric solutions of ten dimensional Einstein’s equation with the only self-dual five-form flux. Then supersymmetry requires  $M_5$  to admit the existence of Killing spinors so that  $M_5$  is Sasaki–Einstein. However, there can be in general other supersymmetric solutions which provide dual field theories still having  $\mathcal{N} = 1$  supersymmetry. Since it is expected that on these backgrounds one has some non-trivial fluxes which contribute to the energy-momentum tensor, the effect of the fluxes should deform the Sasakian structure. Therefore, in order to discover such deformed backgrounds which, if exist, give generalizations of the Sasaki–Einstein manifolds, it might be useful to think how we can deform the Sasakian structure. In fact, Pilch and Warner [9] constructed a non-trivial supersymmetric background  $\text{AdS}_5 \times M_5$ , where  $M_5$  is deformed from  $S^5$  because a non-trivial three-form is present. One interesting approach in this direction is so-called Hitchin’s generalized geometry [10]. By exploiting it, a notion of “generalized Sasaki–Einstein geometry”

which provides general supersymmetric  $\text{AdS}_5$  solutions of type IIB supergravity theory with non-trivial fluxes was introduced and it enables us to study the general structure of the  $\text{AdS}_5/\text{CFT}_4$  correspondence [11, 12]. Unfortunately, however, few explicit examples have been realized.

Our aim is to deform the Sasakian structure by introducing a totally skew-symmetric torsion. The well-known fact is that pseudo-Riemannian manifolds with totally skew-symmetric torsions appear naturally in supergravity theories, where the torsions can be identified with a three-form or other-form fluxes occurring in the theories [13]. On the other hand, many kinds of torsion connections have been studied since old times, e.g., see [14]. Especially in Sasakian geometry, the torsion with which connection preserves the Sasakian structure has been studied for a long time [15–18]. It is known that such a torsion is totally skew-symmetric and is written in terms of the contact one-form:  $T = \eta \wedge d\eta$ . The uniqueness of the torsion was proven in [17]. However, since this kind of torsion connection doesn't deform the Sasakian structure, we need to explore other possibilities.

In this paper, we propose one possible deformation of the Sasakian structure in the presence of totally skew-symmetric torsion. Our idea is the following: A Sasakian manifold is defined as a manifold whose metric cone in one higher dimensions is Kähler. In analogy with this, we demand the cone one dimensional upstairs to be Kähler with torsion. On a Kähler with torsion manifold there exists a unique torsion connection preserving the Hermitian structure, called a *Bismut connection* [19], and the presence of the torsion is deforming the Kähler structure. Thus, Sasakian structure one dimensional downstairs is also deformed as well. As far as we know, this attempt to deform the Sasakian geometry has never been thought and, we believe, it also differs from both the “generalized Sasakian geometry” discussed in [11, 12] and the Sasakian geometry with torsion connection studied in [15–18]. Accordingly, we study general properties of the deformed Sasakian structure.

We also have one other motivation to study this kind of geometry. It was pointed out in [7, 8, 20–22] that some toric Sasaki–Einstein metrics are obtained as the BPS limit of the Kerr-NUT-(A)dS metrics describing higher-dimensional vacuum rotating black holes [23–25]. Recently, it was demonstrated in [26] that the Kerr-Sen black hole solutions [27–29] in heterotic supergravity give rise to Kähler with torsion metrics. Furthermore, we will be able to see in Sec. IV the relationship in five dimensional (un-)gauged supergravity between the Chong-Cvetič-Lü-Pope black hole solution [30] and a certain Euclidean solution admitting the deformed Sasakian structure. Therefore, we are interested in this deformed Sasakian structure as a counterpart of Euclidean solutions which are obtained from higher-dimensional charged rotating black hole solutions.

This paper is organized as follows: In Sec. II, we begin with a brief review of torsion connections. After we define a notion of “Sasaki with torsion structure” in the presence of totally skew-symmetric torsion (see Def. II.1), we look into general properties of the deformed Sasakian structure while clarifying differences from the standard Sasakian structure and introducing some new notions (see Def. II.4). In Sec. III, we present an example of local metrics admitting the deformed Sasakian structure introduced in Sec. II in all odd dimensions, and elaborates curvature properties with respect to the metrics and the cone metrics in one higher dimensions. Hidden symmetry for the metrics is also discussed in this section. In Sec. IV, the solutions of five dimensional minimal (un)-gauged supergravity and eleven dimensional supergravity are obtained. In Sec. V, we discuss the global structure of these solutions briefly. The condition to obtain regular metrics on compact manifolds are argued about the five dimensional minimal gauged supergravity solutions. We study more on the global properties of five dimensional solutions in the special case. In this case, the metric has the enhanced isometry, then it can be regarded as the generalization of  $Y^{p,q}$ . Sec. VI is devoted to summary and discussions. In App. A, we give some calculations which are relevant to the notion introduced by Sec. II. In App. B, the Riemann, Ricci and scalar curvatures for our example of the metrics are computed. We get them with respect to not only the Levi-Civita connection but also the connection with the torsion. In App. C, Calabi-Yau with torsion metrics on the cone are obtained.

## II. DEFORMATION OF SASAKIAN STRUCTURE

In the context of supergravity theories, it is natural to introduce a totally skew-symmetric torsion because it may be identified with three-form field occurring in the theories [13, 14]. Sasakian structure in the presence of torsion has been previously considered by introducing a connection with totally skew-symmetric torsion preserving the Sasakian structure [17, 18]. Given a Sasakian structure, such a connection always uniquely exists and then the torsion is written as  $T = \eta \wedge d\eta$ , where  $\eta$  is the contact one-form. On the other hand, we expect that the existence of torsion no longer preserves the Sasakian geometry because it is caused by the effect of the energy-momentum tensor which changes Einstein’s equation. Therefore, we shall discuss one possible deformation of the Sasakian structure in the presence of totally skew-symmetric torsion.

To fix our notation, we begin with a brief review of a connection with totally skew-symmetric torsion. Let  $(M, g)$  be a Riemannian manifold,  $T$  be a 3-form on  $M$  and  $\{e_a\}$  be an orthonormal

frame on  $TM$ . We define a connection with totally skew-symmetric torsion,  $\nabla^T$ , by

$$g(\nabla_X^T Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2}T(X, Y, Z) , \quad (\text{II.1})$$

where  $X$  and  $Y$  denote vector fields on  $M$  and  $\nabla$  is the Levi-Civita connection of  $g$ . This connection satisfies a metricity condition,  $\nabla^T g = 0$ , and has the same geodesics as  $\nabla$ ,  $\nabla_{\dot{\gamma}}^T \dot{\gamma} = \nabla_{\dot{\gamma}} \dot{\gamma} = 0$  for a geodesic  $\gamma$ . Thus the commutators are related to the Lie brackets by

$$\nabla_X^T Y - \nabla_Y^T X = [X, Y] + T(X, Y) , \quad (\text{II.2})$$

where  $T(X, Y) = \sum_a T(X, Y, e_a)e_a$ . For a  $p$ -form  $\Psi$  a covariant derivative is calculated as

$$\nabla_X^T \Psi = \nabla_X \Psi - \frac{1}{2} \sum_a (X \lrcorner e_a \lrcorner T) \wedge (e_a \lrcorner \Psi) , \quad (\text{II.3})$$

where  $\lrcorner$  represents the inner product. Then, we have

$$\begin{aligned} d^T \Psi &= \sum_a e^a \wedge \nabla_{e_a}^T \Psi \\ &= d\Psi - \sum_a (e_a \lrcorner T) \wedge (e_a \lrcorner \Psi) , \end{aligned} \quad (\text{II.4})$$

$$\begin{aligned} \delta^T \Psi &= - \sum_a e_a \lrcorner \nabla_{e_a}^T \Psi \\ &= \delta\Psi - \frac{1}{2} \sum_{a,b} (e_a \lrcorner e_b \lrcorner T) \wedge (e_a \lrcorner e_b \lrcorner \Psi) , \end{aligned} \quad (\text{II.5})$$

where  $\{e^a\}$  is a dual 1-form frame of  $\{e_a\}$ ,  $e^a(e_b) = \delta^a_b$ .

Suppose  $\mathcal{M}$  is a Hermitian manifold equipped with a complex structure  $J$  and a Hermitian metric  $g$  obeying  $g(X, Y) = g(J(X), J(Y))$  for any vector field  $X$  and  $Y$ . Then it is known that there exists a unique Hermitian connection  $\nabla^B$  with totally skew-symmetric torsion  $B$ , i.e.,  $\nabla^B g = 0$ ,  $\nabla^B J = 0$ . This connection  $\nabla^B$  is known as a *Bismut connection* and the corresponding totally skew-symmetric torsion  $B$  is called a *Bismut torsion* [19], which is written in the form

$$B(X, Y, Z) = d\Omega(J(X), J(Y), J(Z)) , \quad (\text{II.6})$$

where  $\Omega$  is the fundamental 2-form  $\Omega(X, Y) \equiv g(J(X), Y)$ . A Hermitian manifold  $\mathcal{M}$  equipped with the Bismut torsion  $B$  is called a *Kähler with torsion manifold*.

A Riemannian manifold  $(M, g)$  is said to be Sasakian if its metric cone  $(C(M), \bar{g}) = (M \times R_+, \bar{g} = dr^2 + r^2 g)$  is Kähler, and it is demonstrated that the Sasakian structure is derived from the Kähler cone structure, see, e.g., reviews [31–33] and references therein. In analogy with this, we generalize the Sasakian structure to the case when torsion is present as follows:

**Definition II.1** *Let  $(M, g)$  be a Riemannian manifold and  $T$  be a 3-form on  $M$ . Then, we call  $(M, g, T)$  a Sasaki with torsion (ST) manifold if its metric cone  $(C(M), \bar{g})$  is a Kähler with torsion (KT) manifold whose Bismut torsion  $B$  is given by  $B = r^2 T$ .*

Let  $X$  and  $Y$  be vector fields on  $M$ , which can be also viewed as vector fields on  $C(M)$ , and  $\bar{\nabla}^B$  be the Bismut connection associated with the metric cone  $C(M)$ . Then we have the following formulae:

$$\begin{aligned}\bar{\nabla}_{\partial_r}^B \partial_r &= 0, \quad \bar{\nabla}_{\partial_r}^B X = \bar{\nabla}_X^B \partial_r = \frac{1}{r} X, \\ \bar{\nabla}_X^B Y &= \nabla_X^T Y - r g(X, Y) \partial_r,\end{aligned}\tag{II.7}$$

where  $\nabla_X^T Y$  is the connection on  $M$  with totally skew-symmetric torsion  $T$ . Making use of the complex structure  $J$  on the KT cone, we define a vector field  $\xi$  on  $C(M)$  by

$$\xi = J(r \partial_r),\tag{II.8}$$

whose length is given by  $\bar{g}(\xi, \xi) = r^2$ . Since  $\bar{\nabla}_X^B J = 0$  we have

$$\bar{g}(\bar{\nabla}_X^B \xi, Y) = \bar{g}(J(\bar{\nabla}_X^B (r \partial_r)), Y) = \bar{g}(J(X), Y).\tag{II.9}$$

which is anti-symmetric with respect to exchange of  $X$  and  $Y$ . Hence, it is shown that  $\xi$  is a Killing vector field, i.e.,  $\bar{g}(\bar{\nabla}_X^B \xi, Y) + \bar{g}(\bar{\nabla}_Y^B \xi, X) = 0$ . We identify  $M$  with  $M \times \{1\} \subset C(M)$  and thus find that  $\xi$  is a Killing vector field of unit length on  $M$ . We also define on  $M$  a 1-form  $\eta$  and a  $(1, 1)$ -tensor  $\Phi$ , respectively, by

$$\eta(X) = g(\xi, X), \quad \Phi(X) = \nabla_X^T \xi.\tag{II.10}$$

Then we have

$$J(X) = \Phi(X) - r \eta(X) \partial_r.\tag{II.11}$$

From the integrability condition of  $J$ , we now have a condition for the Bismut torsion given by

$$B(X, Y, Z) = B(X, J(Y), J(Z)) + B(J(X), Y, J(Z)) + B(J(X), J(Y), Z).\tag{II.12}$$

Making use of Eq. (II.11) and the relation  $B = r^2 T$ , the condition (II.12) leads to the condition for the torsion  $T$  as

$$T(X, Y, Z) = T(X, \Phi(Y), \Phi(Z)) + T(\Phi(X), Y, \Phi(Z)) + T(\Phi(X), \Phi(Y), Z).\tag{II.13}$$

In the similar fashion to the Sasakian geometry [31–33], we thus have obtained a triple of  $(\xi, \eta, \Phi)$  given by (II.8) and (II.10), and a totally skew-symmetric torsion  $T$  obeying (II.13), which can be interpreted as a generalization of the Sasakian structure. We call it a *Sasaki with torsion (ST) structure on  $M$* . The following proposition provides four equivalent characterizations of the ST structure:

**Proposition II.2** *Let  $(M, g)$  be a Riemannian manifold and  $T$  be a 3-form on  $M$  obeying (II.13). Then the following conditions are equivalent:*

(1)  *$(M, g, T)$  is an ST manifold.*

(2) *There exists a Killing vector field  $\xi$  of unit length on  $M$  so that the dual 1-form  $\eta$  satisfies*

$$\nabla_X^T(d^T\eta) = -2X^\flat \wedge \eta \quad (\text{II.14})$$

*for any vector field  $X$ , where  $X^\flat = g(X, -)$ .*

(3) *There exists a Killing vector field  $\xi$  of unit length on  $M$  so that the  $(1,1)$ -tensor field  $\Phi$  defined by  $\Phi(X) = \nabla_X^T\xi$  satisfies*

$$(\nabla_X^T\Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi \quad (\text{II.15})$$

*for any pair of vector fields  $X, Y$ .*

(4) *There exists a Killing vector field  $\xi$  of unit length on  $M$  so that the curvature satisfies*

$$R^T(X, Y)\xi = g(\xi, Y)X - g(\xi, X)Y + \Phi(T(X, Y)) , \quad (\text{II.16})$$

*for any pair of vector fields  $X, Y$ , where the curvature  $R^T(X, Y)$  is defined by*

$$R^T(X, Y)Z = \nabla_X^T\nabla_Y^TZ - \nabla_Y^T\nabla_X^TZ - \nabla_{[X, Y]}^TZ . \quad (\text{II.17})$$

It is worth mentioning that a  $p$ -form  $\phi$  is called a *special Killing  $p$ -form with torsion* if it satisfies the equations

$$\nabla_X^T\phi = \frac{1}{p+1}X \lrcorner d^T\phi , \quad \nabla_X^T(d^T\phi) = k X \wedge \phi \quad (\text{II.18})$$

with a constant  $k$ . For  $\phi = \eta$ , the first equation in (II.18) implies that its dual vector field  $\xi$  is a Killing vector field. Hence the 1-form  $\eta$  in Prop. II.2 is a special Killing 1-form with torsion. Furthermore, we find that the  $(2\ell + 1)$ -forms

$$\eta^{(\ell)} = \eta \wedge (d^T\eta)^\ell \quad (\text{II.19})$$

for  $\ell = 0, \dots, n$ , are also special Killing forms with torsion on an ST manifold. For general properties of the special Killing forms with torsion, we refer to App. A of [26]. In general, given a special Killing  $p$ -form with torsion  $\phi$  on  $M$ ,

$$\hat{\phi} = r^p dr \wedge \phi + \frac{r^{p+1}}{p+1} d^T \phi \quad (\text{II.20})$$

is a parallel  $(p+1)$ -form on  $C(M)$ , i.e.,  $\bar{\nabla}^B \hat{\phi} = 0$ , see [34]. In particular, for the 1-form  $\eta$  in Prop. II.2 we have a parallel 2-form

$$\hat{\phi} = r dr \wedge \eta + \frac{r^2}{2} d^T \eta = \frac{1}{2} d^T (r^2 \eta), \quad (\text{II.21})$$

which is nothing but a fundamental 2-form  $\Omega = \hat{\phi}$  on  $C(M)$ .

The following statement is an immediate consequence of Prop. II.2.

**Proposition II.3** *Let  $(M, g, T)$  be an ST manifold and  $(\xi, \eta, \Phi)$  be a triple of its ST structure on  $M$ , given in Prop. II.2. Then we have*

$$\eta(\xi) = 1, \quad (\text{II.22})$$

$$\Phi(\Phi(X)) = -X + \eta(X)\xi, \quad (\text{II.23})$$

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y), \quad (\text{II.24})$$

$$\Phi(\xi) = 0, \quad \eta(\Phi(X)) = 0, \quad (\text{II.25})$$

$$N_\Phi(X, Y) + d\eta(X, Y)\xi = 0, \quad (\text{II.26})$$

$$d^T \eta = 2\omega, \quad \xi \lrcorner d\omega = 0, \quad (\text{II.27})$$

where the fundamental 2-form  $\omega$  is defined by  $\omega(X, Y) = g(\Phi(X), Y)$ , and  $N_\Phi$  is the Nijenhuis tensor of type-(1, 2) with respect to  $\Phi$ , given by

$$N_\Phi(X, Y) \equiv [\Phi(X), \Phi(Y)] + \Phi(\Phi([X, Y])) - \Phi([X, \Phi(Y)]) - \Phi([\Phi(X), Y]). \quad (\text{II.28})$$

A Riemannian manifold  $(M, g)$  equipped with a structure  $(\xi, \eta, \Phi)$  satisfying (II.22)–(II.24) is known as an *almost contact metric manifold*. Eq. (II.25) is derived from such a structure, especially Eqs. (II.22) and (II.23). Furthermore, an almost contact metric structure  $(g, \xi, \eta, \Phi)$  is called *normal* if it satisfies (II.26), and a *contact metric structure* if it satisfies  $d\eta = 2\omega$ , respectively (e.g., see [31, 32]). A Sasakian manifold is also known as a normal contact metric manifold. On the other hand, although our ST manifold is an almost normal contact metric manifold, the contact metric structure is deformed by the torsion  $T$  as is seen in (II.27).



**Definition II.4** Let  $(M, g, T)$  be a Riemannian manifold with a 3-form  $T$  obeying (II.13). We call an almost contact metric structure satisfying  $d^T\eta = 2\omega$  and  $\xi \lrcorner d\omega = 0$  a  $T$ -contact metric structure, and call  $(M, g, T, \xi, \eta, \Phi)$  a  $T$ -contact metric manifold. We further call a  $T$ -contact metric manifold a  $TK$ -contact metric manifold if  $\xi$  is a Killing vector field.

The sub-bundle of codimension one,  $\mathcal{D} = \ker \eta \subset TM$  has an almost complex structure defined by  $J_{\mathcal{D}} = \Phi|_{\mathcal{D}}$ . Hence, the sub-bundle  $\mathcal{D}$  together with the endomorphism  $J_{\mathcal{D}}$  provides  $M$  with an almost CR structure of codimension one. The normality condition yields that the almost CR structure is integrable, i.e., the Nijenhuis tensor with respect to  $J_{\mathcal{D}}$  vanishes. Now we show the following proposition.

**Proposition II.5** An ST manifold is a normal  $T$ -contact metric manifold whose torsion  $T_{\mathcal{D}} = T|_{\mathcal{D}}$  is given by a Bismut torsion

$$T_{\mathcal{D}}(X, Y, Z) = d\omega(J_{\mathcal{D}}(X), J_{\mathcal{D}}(Y), J_{\mathcal{D}}(Z)) \quad (\text{II.29})$$

for all  $X, Y, Z \in \mathcal{D}$ .

*Proof.* If  $(M, g, T)$  is an ST manifold, then Prop. II.3 yields that  $M$  is a normal  $T$ -contact metric manifold. Hence, we have  $N^{(i)} = 0$  ( $i = 1, 2$ ), so that (A.1) reduces to

$$\begin{aligned} 2g((\nabla_X^T \Phi)Y, Z) &= -d\omega(X, \Phi Y, \Phi Z) + d\omega(X, Y, Z) + M(X, Y, Z) \\ &\quad + d^T\eta(X, \Phi Z)\eta(Y) - d^T\eta(X, \Phi Y)\eta(Z), \end{aligned} \quad (\text{II.30})$$

Noting that

$$\begin{aligned} d^T\eta(X, \Phi Z)\eta(Y) - d^T\eta(X, \Phi Y)\eta(Z) &= 2\omega(X, \Phi Z)\eta(Y) - 2\omega(X, \Phi Y)\eta(Z) \\ &= 2g(X, Z)g(\xi, Y) - 2g(X, Y)g(\xi, Z) \end{aligned} \quad (\text{II.31})$$

in this equation and using (II.15) we have

$$d\omega(X, \Phi Y, \Phi Z) - d\omega(X, Y, Z) - M(X, Y, Z) = 0. \quad (\text{II.32})$$

When  $X = \xi$  or  $Y, Z = \xi$ , the equation above is automatically satisfied from (II.27) and (A.5). Now it is easy to verify (II.32) is equivalent to (II.29). Conversely, the normality condition  $N^{(1)} = 0$  leads to  $\mathcal{L}_{\xi}\Phi = 0$ , see [32], so that using (A.9) we have a Killing vector field  $\xi$ . It is easy to see that (3) in Prop. II.2 is satisfied.  $\square$

Since an almost contact metric structure is normal if and only if the almost CR structure is integrable and  $\mathcal{L}_{\xi}\Phi = 0$ , see [32], the following proposition immediately follows.

**Proposition II.6** *An ST manifold is a TK-contact metric manifold whose CR structure is integrable and torsion  $T_{\mathcal{D}} = T|_{\mathcal{D}}$  is given by a Bismut torsion.*

Let us close this section by mentioning about some curvature properties. It is known that the Ricci tensor of a Sasakian manifold of dimension  $2n + 1$  is given by  $\text{Ric}(X, \xi) = 2n \eta(X)$ . In the ST manifold case, the Ricci curvature follows from Eq. (II.16) that

$$\begin{aligned} \text{Ric}^T(X, \xi) &= - \sum_a g(R^T(X, e_a)\xi, e_a) \\ &= 2n \eta(X) - \sum_a T(X, e_a, \Phi(e_a)) . \end{aligned} \quad (\text{II.33})$$

### III. SASAKI WITH TORSION METRICS

#### A. Local metrics in all odd dimensions

We shall explicitly present an example of local metrics admitting the deformed Sasakian structure introduced in Sec. II, which we call Sasaki with torsion (ST) metrics. The ST metric in  $2n + 1$  dimensions is given in local coordinates  $(x^a) = (x_\mu, \psi_k)$  where  $\mu = 1, \dots, n$  and  $k = 0, \dots, n$ , by

$$g_{2n+1} = \sum_{\mu=1}^n \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^n Q_\mu \left( \sum_{k=1}^n \sigma_\mu^{(k-1)} d\psi_k \right)^2 + 4 \left( \sum_{k=0}^n \sigma^{(k)} d\psi_k + A \right)^2 , \quad (\text{III.1})$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu} , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^n (x_\mu - x_\nu) , \quad X_\mu = X_\mu(x_\mu) . \quad (\text{III.2})$$

The functions  $\sigma_\mu^{(k)}$  and  $\sigma^{(k)}$  are the  $k$ -th elementary symmetric polynomials in  $x_\mu$ , which are generated by

$$\prod_{\substack{\nu=1 \\ \nu \neq \mu}}^n (\lambda + x_\nu) = \sum_{k=0}^{n-1} \sigma_\mu^{(k)} \lambda^{n-k-1} , \quad \prod_{\nu=1}^n (\lambda + x_\nu) = \sum_{k=0}^n \sigma^{(k)} \lambda^{n-k} , \quad (\text{III.3})$$

and the 1-form  $A$  is given by

$$A = \sum_{\mu=1}^n \frac{N_\mu}{U_\mu} \sum_{k=1}^n \sigma_\mu^{(k-1)} d\psi_k , \quad N_\mu = N_\mu(x_\mu) . \quad (\text{III.4})$$

We should note that the metric (III.1) is an “off-shell” metric, that is, it contains  $2n$  unknown functions  $X_\mu(x_\mu)$  and  $N_\mu(x_\mu)$  depending only on one variable  $x_\mu$ . As we will see in Sec. IV, these

metric functions are determined by considering EOMs of theories. If and only if  $N_\mu$  are given by the polynomials in  $x_\mu$  of the form

$$N_\mu = \sum_{i=0}^{n-1} a_i x_\mu^i, \quad (\text{III.5})$$

where  $a_i$  ( $i = 0, \dots, n-1$ ) are arbitrary constants, the 1-form  $A$  becomes

$$A = \sum_{k=1}^{n-1} (-1)^{k-1} a_{n-k} d\psi_k. \quad (\text{III.6})$$

This is equivalent to taking  $A = 0$  because  $a_i$  are eliminated by gauge transformation of  $d\psi_0$ . The metric (III.1) with  $A = 0$  was studied in [7, 8, 20–22], where it was shown that the metric is obtained as an “off-shell” metric of the BPS limit of the odd-dimensional Kerr-NUT-(A)dS metric and leads to the toric Sasaki-Einstein metrics  $Y^{p,q}$  and  $L^{a,b,c}$  discovered by [4–6]. The metric (III.1) can be also regarded as a Kluza-Klein metric locally describing an  $S^1$ -bundle over  $2n$ -dimensional base space  $B$ ,

$$g = g_B + 4(d\psi_0 + \mathcal{A})^2. \quad (\text{III.7})$$

Although the definition (II.1) implies in general that the metric of  $2n$ -dimensional base space  $(B, g_B)$  is locally KT (see Prop. II.5 and II.6), the present metric is known as an orthotoric Kähler metric established in [35, 36].

Firstly we shall see the conditions in Prop. II.3. To do this, it is convenient to introduce an orthonormal frame  $\{e^a\} = \{e^\mu, e^{\hat{\mu}} = e^{n+\mu}, e^0 = e^{2n+1}\}$  and compute connection 1-forms from the first structure equation

$$de^a + \sum_b \omega^a_b \wedge e^b = 0 \quad (\text{III.8})$$

with  $\omega_{ab} = -\omega_{ba}$ . For the metric (III.1), we can choose the orthonormal frame as

$$e^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad e^{\hat{\mu}} = \sqrt{Q_\mu} \sum_{k=1}^n \sigma_\mu^{(k-1)} d\psi_k, \quad e^0 = 2 \left( \sum_{k=0}^n \sigma^{(k)} d\psi_k + A \right), \quad (\text{III.9})$$

and the connection 1-forms  $\omega^a_b$  are calculated as

$$\omega^\mu_\nu = -\frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e^\mu - \frac{\sqrt{Q_\mu}}{2(x_\mu - x_\nu)} e^\nu, \quad \mu \neq \nu \quad (\text{III.10})$$

$$\omega^\mu_{\hat{\mu}} = -\partial_\mu \sqrt{Q_\mu} e^{\hat{\mu}} + \sum_{\nu \neq \mu} \frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e^{\hat{\nu}} - (1 + \partial_\mu H) e^0, \quad (\text{III.11})$$

$$\omega^\mu_{\hat{\nu}} = \frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e^{\hat{\mu}} - \frac{\sqrt{Q_\mu}}{2(x_\mu - x_\nu)} e^{\hat{\nu}}, \quad \mu \neq \nu \quad (\text{III.12})$$

$$\omega^{\hat{\mu}}_{\hat{\nu}} = -\frac{\sqrt{Q_{\nu}}}{2(x_{\mu} - x_{\nu})} e^{\mu} - \frac{\sqrt{Q_{\mu}}}{2(x_{\mu} - x_{\nu})} e^{\nu} , \quad \mu \neq \nu \quad (\text{III.13})$$

$$\omega^{\mu}_0 = -(1 + \partial_{\mu} H) e^{\hat{\mu}} , \quad (\text{III.14})$$

$$\omega^{\hat{\mu}}_0 = (1 + \partial_{\mu} H) e^{\mu} , \quad (\text{III.15})$$

where  $H$  is defined by

$$H = \sum_{\mu=1}^n \frac{N_{\mu}}{U_{\mu}} . \quad (\text{III.16})$$

We introduce a 1-form  $\eta = e^0$ , vector field  $\xi = e_0$  and  $(1,1)$ -tensor  $\Phi$  in Prop. II.3 as

$$\Phi(e_{\mu}) = e_{\hat{\mu}} , \quad \Phi(e_{\hat{\mu}}) = -e_{\mu} , \quad \Phi(e_0) = 0 . \quad (\text{III.17})$$

For the triple  $(\xi, \eta, \Phi)$  together with the metric  $g$ , the conditions (II.22)–(II.24) hold. Namely, the set  $(g, \xi, \eta, \Phi)$  is an almost contact metric structure on  $M$ . Moreover, by using relations  $\nabla_{e_a} e^b(e_c) = -\omega^b_c(e_a)$  we can compute the covariant derivatives as (B.1)–(B.13) in App. B, and the commutation relations  $[e_a, e_b] \equiv \nabla_{e_a} e_b - \nabla_{e_b} e_a$  are obtained. Eq. (II.26) is demonstrated from the obtained commutation relations, and implies that the almost contact metric structure is normal. However, we find that  $\eta$  is not in general a contact 1-form because we have

$$d\eta = 2 \sum_{\mu=1}^n (1 + \partial_{\mu} H) e^{\mu} \wedge e^{\hat{\mu}} , \quad (\text{III.18})$$

and hence there is a possibility that  $\eta \wedge (d\eta)^n = 0$  at some points on the manifold. If  $H$  is constant, we have  $d\eta(X, Y) = 2g(\Phi(X), Y)$  so that  $\eta$  is a contact 1-form. We further find that this example is a quasi-Sasakian metric [37], i.e.,  $d\omega = 0$  where  $\omega$  is the fundamental 2-form given by  $\omega(X, Y) = g(\Phi(X), Y)$ . In fact, we have

$$\omega = \sum_{\mu=1}^n e^{\mu} \wedge e^{\hat{\mu}} = d \left[ \sum_{k=0}^n \sigma^{(k)} d\psi_k \right] . \quad (\text{III.19})$$

Next, let us see the conditions in Prop. II.2. We introduce the torsion  $T$  and compute the covariant derivatives with respect to the torsion connection  $\nabla^T$ . Since the torsion  $T$  satisfying Eq. (II.27) is given by

$$T = 2 \sum_{\mu=1}^n \partial_{\mu} H e^{\mu} \wedge e^{\hat{\mu}} \wedge e^0 , \quad (\text{III.20})$$

we can check that Eq. (II.13) holds. We emphasize again that the torsion (III.20) differs from the torsion preserving the Sasakian structure,  $\eta \wedge d\eta$ , discussed in [17]. Namely,  $\nabla^T \xi \neq 0$  and

$\nabla^T \Phi \neq 0$ . The covariant derivatives  $\nabla_{e_a}^T e_b$  are calculated as (B.29)–(B.41) in App. B. Using these expressions, we find that

$$\nabla_X^T \xi = \Phi(X) . \quad (\text{III.21})$$

It is also shown that for any vector field  $X$ ,

$$\nabla_X^T \eta = \frac{1}{2} X \lrcorner d^T \eta , \quad \nabla_X^T (d^T \eta) = -2X \wedge \eta , \quad (\text{III.22})$$

which proves Eq. (II.18) with  $k = -2$  so that  $\eta$  is a special Killing 1-form with torsion.

## B. The cone metrics

Let us consider the Riemannian cone of the metric (III.1),

$$\bar{g} = dr^2 + r^2 g_{2n+1} . \quad (\text{III.23})$$

For the cone metric  $\bar{g}$ , we introduce an orthonormal basis  $\{\bar{e}^\alpha\}$  as

$$\bar{e}^r = dr , \quad \bar{e}^a = r e^a . \quad (\text{III.24})$$

The connection 1-forms  $\bar{\omega}^\alpha{}_\beta$  with respect to  $\bar{g}$  are calculated as

$$\bar{\omega}^r{}_a = -\frac{1}{r} \bar{e}^a , \quad \bar{\omega}^a{}_b = \omega^a{}_b , \quad (\text{III.25})$$

where  $\omega^a{}_b$  is given by (III.10)–(III.15), and the commutation relations  $[\bar{e}_\alpha, \bar{e}_\beta]$  are calculated in the similar manner to previous section. We introduce an almost complex structure  $J$  by

$$J(\bar{e}_r) = \bar{e}_0 , \quad J(\bar{e}_0) = -\bar{e}_r , \quad J(\bar{e}_\mu) = \bar{e}_{\hat{\mu}} , \quad J(\bar{e}_{\hat{\mu}}) = -\bar{e}_\mu . \quad (\text{III.26})$$

Then it is directly checked that for the almost complex structure  $J$ , the Nijenhuis tensor vanishes so that  $J$  is integrable, and the cone metric  $\bar{g}$  is Hermitian,

$$\bar{g}(X, Y) = \bar{g}(J(X), J(Y)) . \quad (\text{III.27})$$

The fundamental form  $\Omega(X, Y) = \bar{g}(J(X), Y)$  can be written as

$$\Omega = \bar{e}^r \wedge \bar{e}^0 + \sum_{\mu=1}^n \bar{e}^\mu \wedge \bar{e}^{\hat{\mu}} = \frac{1}{2} d^T (r^2 e^0) . \quad (\text{III.28})$$

Since  $(M, g, J)$  is a Hermitian manifold, there exists the Bismut connection, a unique Hermitian connection  $\bar{\nabla}^B$  with totally skew-symmetric torsion  $B$ . From (II.6) the Bismut torsion is explicitly obtained as

$$B = \sum_{\mu=1}^n \frac{2}{r} \partial_{\mu} H \bar{e}^{\mu} \wedge \bar{e}^{\hat{\mu}} \wedge \bar{e}^0 = r^2 T , \quad (\text{III.29})$$

where  $T$  is given by (III.20). We finally note that the Killing vector fields  $\partial/\partial\psi_k$  ( $k = 0, 1, \dots, n$ ) preserve the KT structure on the cone,

$$\mathcal{L}_{\partial_k} \Omega = 0, \quad \mathcal{L}_{\partial_k} B = 0. \quad (\text{III.30})$$

### C. Hidden symmetry

Sasakian geometry is relevant to Killing-Yano symmetry. It was shown in [15, 16] that a Sasakian manifold of  $2n + 1$  dimensions has rank- $(2p + 1)$  special Killing forms in the form  $\eta \wedge (d\eta)^p$  ( $0 \leq p \leq n$ ). Killing-Yano symmetry was originally defined as Killing-Yano (KY) tensors [38] and conformal Killing-Yano (CKY) tensors [34, 39, 40] from a purely mathematical view point, and later has played an important role in the study of black hole physics. One of the features is that general metrics admitting a rank-2 closed CKY tensor were obtained in four [41, 42] and higher [43–46] dimensions, and such metrics allow us to have remarkable properties in mathematical physics: For instance, separation of variables for the Hamilton-Jacobi, Klein-Gordon and Dirac equations. Since many higher-dimensional vacuum solutions of Einstein's equation with cosmological constant describing rotating black holes with spherical horizon topology [23–25] are covered in this class of the metrics, they possess similar integrability structures due to Killing-Yano symmetry, e.g., see reviews [47–49].

By considering totally skew-symmetric torsions, Killing-Yano symmetry is naturally generalized. A *generalized conformal Killing-Yano (GCKY) tensor*  $k$  with respect to torsion  $T$  was introduced by [50] as a  $p$ -form satisfying for any vector field  $X$  and a 3-form  $T$ ,

$$\nabla_X^T k = \frac{1}{p+1} X \lrcorner d^T k - \frac{1}{D-p+1} X^{\flat} \wedge \delta^T k , \quad (\text{III.31})$$

where  $X^{\flat}$  is a dual 1-form of  $X$ . In particular, we call a GCKY tensor  $f$  obeying  $\delta^T f = 0$  a *generalized Killing-Yano (GKY) tensor*, and a GCKY tensor  $h$  obeying  $d^T h = 0$  a *generalized closed conformal Killing-Yano (GCCKY) tensor*. It is known that the generalized Killing-Yano symmetry occurs in five-dimensional minimal supergravity [50], abelian heterotic supergravity and its higher-dimensional generalization [51]. General properties of such a symmetry [51–53] and general metrics

admitting a rank-2 GCCKY tensor have been studied in recent years [26]. Furthermore, it has been clarified by [26, 54, 56, 57] that geometry with the generalized Killing–Yano symmetry is related to Kähler geometry studied by [35, 36] and Sasakian geometry, which are obtained as a BPS limit of Euclideanized higher-dimensional black hole spacetimes.

As we have seen in previous sections, the metric (III.1) can be regarded as a natural generalization of Sasakian metrics in the presence of torsion. Thus it is natural to expect the existence of the generalized Killing–Yano symmetry. We also find that for the metric (III.1), the Hamilton-Jacobi equation for geodesics

$$\partial_\lambda S + g^{ab}(\partial_a S)(\partial_b S) = 0 \quad (\text{III.32})$$

can be solved by separation of variables. This implies the existence of not only Killing vector fields  $\partial/\partial\psi_k$ , but rank-2 Killing–Stäckel tensors obeying

$$\nabla_{(a} K_{bc)} = 0, \quad K_{(ab)} = K_{ab}. \quad (\text{III.33})$$

In general, given a rank- $p$  GKY tensor  $f$ , one can always generate a rank-2 Killing–Stäckel tensor  $K$  by

$$K_{ab} = f_{ac_1 \dots c_{p-1}} f_b{}^{c_1 \dots c_{p-1}}. \quad (\text{III.34})$$

Therefore, it is worth asking whether the metric (III.1) admits the existence of GKY tensors with respect to a torsion.

Exploring GKY tensors gives rise to the problem what the torsion is. The natural torsion is the 3-form  $T$  related to the ST structure, given by (III.20). Since the first equation in (II.18) is same as the GKY equation, a special Killing  $p$ -form with torsion is alternatively said to be a *rank- $p$  special GKY tensor*. As was already seen in (II.19),  $\eta^{(\ell)} \equiv \eta \wedge (d^T \eta)^\ell$  for  $\ell = 0, \dots, n$  are rank- $(2\ell + 1)$  special GKY tensors with respect to torsion  $T$ . Thus we have  $n + 1$  GKY tensors. However, these GKY tensors  $\eta^{(\ell)}$  do not give rise to non-trivial rank-2 Killing tensors. In fact, every GKY tensor generates the only metric essentially.

On the other hand, for the metric (III.1) we can find other GKY tensors  $f^{(j)}$ , which are not special, by introducing another torsion. We introduce a 2-form  $\hat{h}$  and 3-form  $G$  as

$$\hat{h} = \sum_{\mu=1} \sqrt{x_\mu} e^\mu \wedge e^{\hat{\mu}}, \quad (\text{III.35})$$

$$G = \sum_{\mu \neq \nu} \frac{1}{\sqrt{x_\mu} + \sqrt{x_\nu}} \sqrt{\frac{Q_\nu}{x_\nu}} e^\mu \wedge e^{\hat{\mu}} \wedge e^{\hat{\nu}} + \sum_{\mu=1}^n 2(1 + \partial_\mu H) e^\mu \wedge e^{\hat{\mu}} \wedge e^0. \quad (\text{III.36})$$

Then it is demonstrated that for the metric (III.1), the  $(2j+1)$ -forms

$$h^{(\ell)} \equiv e^0 \wedge (\hat{h})^j = e^0 \wedge \hat{h} \wedge \cdots \wedge \hat{h} \quad (\text{III.37})$$

for  $j = 1, \dots, n$ , are rank- $(2j+1)$  GCCKY tensors with respect to torsion  $G$ , obeying for any vector field  $X$

$$\nabla_X^G h^{(j)} = -\frac{1}{D-2j} X^\flat \wedge \delta^G h^{(j)} . \quad (\text{III.38})$$

From general properties of GCKY tensors (e.g., see [51]), GCCKY tensors  $h^{(j)}$  generate GKY tensors  $f^{(j)}$  by  $f^{(j)} = *h^{(j)}$ . These GKY tensors  $f^{(j)}$  generate rank-2 Killing tensors  $K^{(j)}$  by  $K_{ab}^{(j)} = [\ell!^2(n-2j-1)!]^{-1} f_{ac_1 \dots c_{D-2\ell-2}}^{(j)} f_{b \quad c_1 \dots c_{D-2\ell-2}}^{(j)}$ , which are explicitly written as

$$K^{(j)} = \sum_{\mu=1}^n \sigma_\mu^{(j)} (e^\mu \otimes e^\mu + e^{\hat{\mu}} \otimes e^{\hat{\mu}}) . \quad (\text{III.39})$$

## IV. SUPERGRAVITY SOLUTIONS

### A. Five-dimensional minimal (un-)gauged supergravity

In this section we investigate the Euclidean solutions of supergravity theories, which are written in the form (III.1). We consider the five-dimensional minimal gauged supergravity. The action is given by

$$\mathcal{L}_5 = *(\mathcal{R} - \Lambda) - \frac{1}{2} F_{(2)} \wedge *F_{(2)} + \frac{1}{3\sqrt{3}} F_{(2)} \wedge F_{(2)} \wedge A_{(1)} , \quad (\text{IV.1})$$

where  $F_{(2)} = dA_{(1)}$  is a 2-form field strength of a Maxwell field  $A_{(1)}$ . The equations of motion are

$$R_{ab} = -4g_{ab} + \frac{1}{2} \left( F_{(2)ac} F_{(2)b}{}^c - \frac{1}{6} g_{ab} F_{(2)cd} F_{(2)}^{cd} \right) , \quad (\text{IV.2})$$

$$d * F_{(2)} - \frac{1}{\sqrt{3}} F_{(2)} \wedge F_{(2)} = 0 , \quad (\text{IV.3})$$

where the cosmological constant is normalized as  $\Lambda = -12$ .

We should note that for the Euclidean solutions, we must consider the Euclidean action that is obtained by Wick rotation. Since this corresponds to changing the sign of the whole right-hand side in Eq. (IV.2), the cosmological constant can be interpreted as positive. Actually, we investigate the solutions of the Einstein equation for Euclidean signature,

$$R_{ab} = 4g_{ab} - \frac{1}{2} \left( F_{(2)ac} F_{(2)b}{}^c - \frac{1}{6} g_{ab} F_{(2)cd} F_{(2)}^{cd} \right) . \quad (\text{IV.4})$$



As we will see later, our Wick rotation is achieved by making the fiber direction of solutions timelike, so as to satisfy the original Einstein equation (IV.2). We will also find that this transformation does not break the reality of the matter flux.

We assume the form of the gauge potential  $A_{(1)}$  and the metric functions  $N_\mu$  so as to solve the Maxwell Chern-Simon equation (IV.3),

$$A_{(1)} = c_F \sum_{\mu=1}^2 \frac{q_\mu}{U^\mu} \sum_{k=1}^2 \sigma_\mu^{(k-1)} d\psi_k , \quad (\text{IV.5})$$

$$N_\mu = a_1 x_\mu + q_\mu , \quad (\text{IV.6})$$

with  $c_F$ ,  $a_1$  and  $q_\mu$  constant parameters. Then the field strength is

$$F_{(2)} = c_F \left( \partial_1 H e^1 \wedge e^{\hat{1}} + \partial_2 H e^2 \wedge e^{\hat{2}} \right) , \quad (\text{IV.7})$$

where we have  $\partial_1 H = -\partial_2 H$ . Since  $F_{(2)}$  is closed and has the following properties

$$*F_{(2)} = -F_{(2)} \wedge \eta , \quad F_{(2)} \wedge \omega = 0 , \quad (\text{IV.8})$$

where  $\omega(X, Y) = g(\Phi(X), Y)$ , Eq. (IV.3) can be solved easily

$$\begin{aligned} d * F_{(2)} &= -F_{(2)} \wedge d\eta \\ &= -\frac{2}{c_F} F_{(2)} \wedge F_{(2)} . \end{aligned} \quad (\text{IV.9})$$

Therefore the constant  $c_F$  is determined as  $c_F = -2\sqrt{3}$ . Eq. (IV.4) requires that  $X_\mu(x_\mu)$  takes the form

$$X_\mu = -4x_\mu^3 + \sum_{i=1}^2 c_i x_\mu^i + b_\mu - 8q_\mu x_\mu , \quad (\text{IV.10})$$

where  $c_i$ ,  $b_\mu$  and  $q_\mu$  are constants.

This metric admits our torsion 3-form  $T$ , which preserves the KT structure of the cone. Note that the torsion can be constructed from the matter  $F_{(2)}$ . It is written as

$$T = \frac{1}{\sqrt{3}} * F_{(2)} . \quad (\text{IV.11})$$

In contrast, the torsion preserving the almost metric contact structure needs the additional term  $\eta \wedge \omega$ . These solutions have a certain correspondence to the black hole solutions constructed by [30]. The black holes exist on the same theory, minimal gauged supergravity, and admit the similar torsion which can be written in the same form by their matter flux. It was founded that this torsion constructs a generalized closed conformal Killing Yano 2-form on the black holes [50]. However,

the hidden symmetry exists on our solutions in the different form; the GCCKY tensors  $h^{(j)}$  are odd rank tensors with the torsion  $G$ , which cannot be written only by the matter flux.

We found the un-gauged minimal supergravity solutions in the similar way. The solutions are provided when (III.1) and (IV.5) take the form

$$X_\mu = \sum_{i=1}^2 c_i x_\mu^i + b_\mu, \quad N_\mu = -x_\mu^2 + a_1 x_\mu + q_\mu. \quad (\text{IV.12})$$

The solutions can be changed into Lorentz signature as in the case of the gauged supergravity solutions. In the un-gauged case, Wick rotation changes only the metric in the form

$$g_L = \sum_{\mu=1}^2 \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^2 Q_\mu \left( \sum_{k=1}^2 \sigma_\mu^{(k-1)} d\psi_k \right)^2 - 4 \left( \sum_{k=0}^2 \sigma^{(k)} d\psi_k + A \right)^2. \quad (\text{IV.13})$$

The gauged minimal supergravity solutions need to correct  $X_\mu$  as

$$X_\mu = 4x_\mu^3 + \sum_{i=1}^2 c_i x_\mu^i + b_\mu + 8q_\mu x_\mu. \quad (\text{IV.14})$$

This arises from the negativity of the cosmological constant. In both cases, the vector potential remains the form as (IV.5).

## B. Eleven-dimensional supergravity

We consider the eleven-dimensional supergravity. The action is given by

$$\mathcal{L}_{11} = *\mathcal{R} - \frac{1}{2} F_{(4)} \wedge *F_{(4)} + \frac{1}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)} \quad (\text{IV.15})$$

where  $F_{(4)} = dA_{(3)}$  is a 4-form flux of a 3-form gauge potential  $A_{(3)}$ . The equations of motion are

$$R_{ab} = \frac{1}{12} \left( F_{(4)acde} F_{(4)b}{}^{cde} - \frac{1}{12} g_{ab} F_{(4)abcd} F_{(4)}^{abcd} \right), \quad (\text{IV.16})$$

$$d * F_{(4)} - \frac{1}{2} F_{(4)} \wedge F_{(4)} = 0. \quad (\text{IV.17})$$

As is the five-dimensional case, we examine the Euclidean solutions satisfying the Einstein equation which are obtained by changing the sign of the right-hand side in Eq. (IV.16).

We assume that the field strength  $F_{(4)}$  takes the form

$$F_{(4)} = \frac{1}{2} \sum_{\mu \neq \nu} F_{\mu\nu} e^\mu \wedge e^{\hat{\mu}} \wedge e^\nu \wedge e^{\hat{\nu}}, \quad (\text{IV.18})$$

where

$$F_{\mu\nu} = 2\ell_1 + \ell_2 (\partial_\mu H + \partial_\nu H), \quad (\text{IV.19})$$

and  $H$  is still given by (III.16) and  $\ell_1, \ell_2$  are constant. Under this assumption, the field strength becomes closed,  $dF_{(4)} = 0$ , and the co-derivative is given by

$$\begin{aligned} \delta F_{(4)} = & - \sum_{\mu \neq \nu} \sqrt{Q_\mu} \left( \partial_\mu F_{\mu\nu} + \sum_{\rho \neq \mu, \nu} \frac{F_{\mu\nu} - F_{\rho\nu}}{x_\mu - x_\rho} \right) e^{\hat{\mu}} \wedge e^\nu \wedge e^{\hat{\nu}} \\ & + 2 \sum_{\mu \neq \nu} F_{\mu\nu} (1 + \partial_\mu H) e^\nu \wedge e^{\hat{\nu}} \wedge e^0 . \end{aligned} \quad (\text{IV.20})$$

Substituting the expressions (IV.18) and (IV.20) into Eq. (IV.17), we obtain  $\ell_1 = \ell_2 = -2$  and

$$N_\mu = -x_\mu^5 + \sum_{i=1}^4 a_i x_\mu^i + q_\mu . \quad (\text{IV.21})$$

Then we have

$$F_{(4)} = -2 \sum_{\mu \neq \nu} (1 + \partial_\mu H) e^\mu \wedge e^{\hat{\mu}} \wedge e^\nu \wedge e^{\hat{\nu}} . \quad (\text{IV.22})$$

The Einstein equation (IV.16) reduces to

$$\partial_\mu^2 Q_T - 4 \sum_{\nu \neq \mu} K_{\mu\nu} = 0 , \quad (\text{IV.23})$$

where

$$K_{\mu\nu} \equiv -\frac{1}{4} \frac{\partial_\mu Q_T}{x_\mu - x_\nu} + \frac{1}{4} \frac{\partial_\nu Q_T}{x_\mu - x_\nu} , \quad Q_T \equiv \sum_{\mu=1}^n Q_\mu . \quad (\text{IV.24})$$

This equation can be solved by

$$X_\mu = \sum_{i=1}^5 c_i x_\mu^i + b_\mu \quad (\text{IV.25})$$

with free parameters  $c_i$  and  $b_\mu$ .

## V. GLOBAL ANALYSIS

### A. Compact manifolds in five dimensions

We briefly argue about the global structure of the five-dimensional minimal gauged supergravity solution obtained in Sec. IV. Our aim in this section is to construct regular metrics on compact manifolds.

1. Generalization of  $L^{a,b,c}$

The metric is written in the form

$$g_5 = \frac{x-y}{X} dx^2 + \frac{y-x}{Y} dy^2 + \frac{X}{x-y} (d\psi_1 + y d\psi_2)^2 + \frac{Y}{y-x} (d\psi_1 + x d\psi_2)^2 \\ + 4 \left( d\psi_0 + (x+y) d\psi_1 + xy d\psi_2 + \frac{q_1 - q_2}{x-y} d\psi_1 + \frac{q_1 y - q_2 x}{x-y} d\psi_2 \right)^2, \quad (\text{V.1})$$

where

$$X = -4x(x - \alpha_1)(x - \alpha_2) + b_1 - 8q_1 x, \quad Y = -4y(y - \alpha_1)(y - \alpha_2) + b_2 - 8q_2 y. \quad (\text{V.2})$$

and  $\alpha_i$  ( $i = 1, 2$ ),  $b_\mu$  and  $q_\mu$  ( $\mu = 1, 2$ ) are free parameters. However, not all the parameters are non-trivial. There is a scaling symmetry of the metric, under which we take

$$x_\mu \rightarrow \lambda x_\mu, \quad \psi_k \rightarrow \lambda^{-k} \psi_k, \\ \alpha_i \rightarrow \lambda \alpha_i, \quad b_\mu \rightarrow \lambda^3 b_\mu, \quad q_i \rightarrow \lambda^2 q_\mu. \quad (\text{V.3})$$

The metric also has a shift symmetry which is taken by

$$x_\mu \rightarrow x_\mu + \lambda, \quad \psi_0 \rightarrow \psi_0 - \lambda^2 \psi_2, \quad \psi_1 \rightarrow \psi_1 - \lambda \psi_2, \\ \alpha_1 + \alpha_2 \rightarrow \alpha_1 + \alpha_2 - 3\lambda, \quad \alpha_1 \alpha_2 + 2q_\mu \rightarrow \alpha_1 \alpha_2 + 2q_\mu - 2(\alpha_1 + \alpha_2)\lambda + 3\lambda^2, \\ b_\mu \rightarrow b_\mu - 4(\alpha_1 \alpha_2 + 2q_\mu)\lambda + 4\lambda^2 - 4\lambda^3. \quad (\text{V.4})$$

In order to obtain regular metrics on compact manifolds, we must impose appropriate regions of the coordinates which correspond to making an appropriate choice of the parameters. Let us assume that  $x_i$  and  $y_i$  ( $i = 1, 2, 3$ ) are real roots of the equations  $X(x) = 0$  and  $Y(y) = 0$ , and they are satisfying the inequalities  $x_1 < x_2 < x_3$  and  $y_1 < y_2 < y_3$ . If we choose the region of the coordinates as  $x_1 \leq x \leq x_2 < y_2 \leq y \leq y_3$ , then the metric is positive definite, except for the boundaries  $x = x_1$  and  $x_2$  as well as  $y = y_2$  and  $y_3$ . From the relationship between the coefficients and solutions, we have

$$\alpha_1 + \alpha_2 = x_1 + x_2 + x_3 = y_1 + y_2 + y_3, \quad (\text{V.5})$$

$$\alpha_1 \alpha_2 + 2q_1 = x_1 x_2 + x_1 x_3 + x_2 x_3, \quad \alpha_1 \alpha_2 + 2q_2 = y_1 y_2 + y_1 y_3 + y_2 y_3, \quad (\text{V.6})$$

$$b_1 = 4x_1 x_2 x_3, \quad b_2 = 4y_1 y_2 y_3. \quad (\text{V.7})$$

Following [7, 8], we can extend the metric smoothly onto the boundaries: Since  $\partial/\partial\psi_0$ ,  $\partial/\partial\psi_1$  and  $\partial/\partial\psi_2$  are linearly independent Killing vector fields, the general Killing vector field is written

as

$$v = \sum_{k=0}^2 \omega_k \frac{\partial}{\partial \psi_k} , \quad (\text{V.8})$$

where  $\omega_k$  are constants. The length of  $v$  is given by

$$\begin{aligned} v^2 = & \frac{X(x)}{x-y} (\omega_1 + y\omega_2)^2 + \frac{Y(y)}{y-x} (\omega_1 + x\omega_2)^2 \\ & + 4 \left( \omega_0 + (x+y)\omega_1 + xy\omega_2 + \frac{q_1 - q_2}{x-y} \omega_1 + \frac{q_1 y - q_2 x}{x-y} \omega_2 \right)^2 . \end{aligned} \quad (\text{V.9})$$

Using this expression, we can construct the associated normalized Killing vector fields  $v_i$  ( $i = 1, 2$ ) and  $\ell_j$  ( $j = 2, 3$ ) such that their lengths are vanishing at the corresponding boundaries  $x = x_i$  and  $y = y_j$ , which are given by

$$\begin{aligned} v_i &= \frac{2}{X'(x_i)} \left( (q_1 + x_i^2) \frac{\partial}{\partial \psi_0} - x_i \frac{\partial}{\partial \psi_1} + \frac{\partial}{\partial \psi_2} \right) , \\ \ell_j &= \frac{2}{Y'(y_j)} \left( (q_2 + y_j^2) \frac{\partial}{\partial \psi_0} - y_j \frac{\partial}{\partial \psi_1} + \frac{\partial}{\partial \psi_2} \right) . \end{aligned} \quad (\text{V.10})$$

Their normalizations are taken in such a way that the surface gravity is equal to unity,

$$\frac{g^{ab}(\partial_a v_i^2)(\partial_b v_i^2)}{4v_i^2} \Big|_{x=x_i} = \frac{g^{ab}(\partial_a \ell_j^2)(\partial_b \ell_j^2)}{4\ell_j^2} \Big|_{y=y_j} = 1 . \quad (\text{V.11})$$

Thus the metric extends smoothly onto the boundaries if Killing vector fields  $v_i$  and  $\ell_j$  have period  $2\pi$ .

Since we have four vector fields  $v_i$  and  $\ell_j$ , they must satisfy a linear relation

$$n_1 v_1 + n_2 v_2 + m_1 \ell_2 + m_2 \ell_3 = 0 \quad (\text{V.12})$$

for integral coefficients  $(n_1, n_2, m_1, m_2)$ , which are assumed to be coprime. To avoid conical singularities, any three of the integers must be also coprime. Substituting (V.10) into Eq. (V.12), it can be solved as

$$\begin{aligned} \frac{n_1}{(x_3 - x_1)[q + (x_2 - y_2)(x_2 - y_3)]} &= \frac{n_2}{(x_3 - x_2)[q + (x_1 - y_2)(x_1 - y_3)]} \\ &= \frac{m_1}{(y_2 - y_1)[q - (x_1 - y_3)(x_2 - y_3)]} = \frac{m_2}{(y_3 - y_1)[q - (x_1 - y_2)(x_2 - y_2)]} , \end{aligned} \quad (\text{V.13})$$

where

$$q \equiv q_1 - q_2 = \frac{x_1 x_2 + x_1 x_3 + x_2 x_3 - y_1 y_2 - y_1 y_3 - y_2 y_3}{2} . \quad (\text{V.14})$$

Since we have degrees of freedom under the scaling symmetry (V.3) and the shift symmetry (V.4), the value of (V.13) can be set to 1 and we can take  $b_2 = 0$  without loss of generality. Then we have  $y_1 = 0$  and Eq. (V.13) leads to

$$n_1 = (x_3 - x_1)[q + (x_2 - y_2)(x_2 - y_3)] , \quad (\text{V.15})$$

$$n_2 = (x_3 - x_2)[q + (x_1 - y_2)(x_1 - y_3)] , \quad (\text{V.16})$$

$$m_1 = y_2[q - (x_1 - y_3)(x_2 - y_3)] , \quad (\text{V.17})$$

$$m_2 = y_3[q - (x_1 - y_2)(x_2 - y_2)] , \quad (\text{V.18})$$

where

$$q = \frac{x_1x_2 + x_1x_3 + x_2x_3 - y_2y_3}{2} . \quad (\text{V.19})$$

Thus the problem of constructing regular metrics on compact manifolds results in solving four coupled algebraic equations (V.15)–(V.18) for a set of coprime integers  $(n_1, n_2, m_1, m_2)$ , under the conditions  $x_3 = y_2 + y_3 - x_1 - x_2$ ,  $x_1 < x_2 < x_3$ ,  $0 < y_2 < y_3$  and  $x_2 < y_2$  for the real roots  $x_i$  and  $y_i$ . In particular, when we take  $q = 0$ , Eqs. (V.15)–(V.18) give rise to the condition

$$n_1 + n_2 + m_1 + m_2 = 0 , \quad (\text{V.20})$$

which leads to the toric Sasaki–Einstein metrics  $L^{n_1, n_2, m_1}$  on  $S^2 \times S^3$ , discussed in [6–8]. When the parameter  $q$  is non-zero, we find Sasaki with torsion metrics  $L^{n_1, n_2, m_1, m_2}$  parameterized by independent four integers.

$n_1$	$n_2$	$m_1$	$m_2$	$x_1$	$x_2$	$x_3$	$y_1$	$y_2$	$y_3$	$q$
-4	-3	-1	-2	-1.32023	-1.25127	3.11486	0	0.167499	0.375858	-3.21042
-4	-2	-1	-3	-1.27727	-1.14007	3.17205	0	0.155888	0.59882	-3.15254
-4	-1	-2	-3	-1.25653	-1.04966	3.20068	0	0.329791	0.564696	-3.12434
-4	1	-2	-3	-1.18938	-0.78852	3.0232	0	0.372652	0.672647	-2.6462
-4	2	-1	-3	-1.11468	-0.543466	2.82041	0	0.188869	0.9734	-2.12735
-4	3	-1	-2	-1.11385	-0.202506	2.46249	0	0.263875	0.882254	-1.62438
-3	-2	-1	-4	-1.15876	-1.09101	3.24621	0	0.142997	0.853444	-3.08052
-3	-1	-2	-4	-1.14654	-1.00989	3.26472	0	0.302705	0.805588	-3.06305
-3	1	-2	-4	-1.05629	-0.741916	3.11358	0	0.3276	0.987783	-2.56939
-3	2	-1	-4	-0.899549	-0.461233	3.04032	0	0.146529	1.53301	-1.97347
-2	-1	-3	-4	-1.06256	-0.9939	3.29153	0	0.483516	0.751555	-3.0381
-2	1	-3	-4	-0.969754	-0.731764	3.13442	0	0.536657	0.896244	-2.55231
-1	2	-3	-4	-0.761789	-0.47932	2.987	0	0.578562	1.16732	-2.00871
-1	4	-3	-2	-0.358631	0.309102	2.83766	0	0.50795	2.28018	-0.704807
1	4	-3	-2	0.167966	1.36545	2.52543	0	1.90031	2.15854	0
2	1	-3	4	1.58023	2.19861	2.46249	0	2.66499	3.57634	1.62438
2	1	3	4	2.739	2.94736	3.11486	0	4.36613	4.43508	3.21042
2	3	-4	-1	0.36689	0.625118	2.52543	0	1.15997	2.35746	0
2	3	-4	1	0.557479	2.32971	2.83766	0	2.52855	3.19629	0.704807
3	1	-2	4	1.84701	2.63154	2.82041	0	3.36388	3.93509	2.12735
3	1	2	4	2.57323	3.01616	3.17205	0	4.31211	4.44931	3.15254
3	2	-1	4	2.35055	2.65055	3.0232	0	3.81172	4.21258	2.6462
3	2	1	4	2.63598	2.87089	3.20068	0	4.25033	4.45721	3.12434
4	1	-2	3	1.50731	2.8938	3.04032	0	3.50156	3.93987	1.97347
4	1	2	3	2.39276	3.10321	3.24621	0	4.33722	4.40497	3.08052
4	2	-1	3	2.1258	2.78598	3.11358	0	3.8555	4.16987	2.56939
4	2	1	3	2.45913	2.96201	3.26472	0	4.2746	4.41126	3.06305
4	3	-2	1	1.81967	2.40843	2.987	0	3.46631	3.74878	2.00871
4	3	-1	2	2.23817	2.59776	3.13442	0	3.86618	4.10417	2.55231
4	3	1	2	2.53997	2.80801	3.29153	0	4.28543	4.35409	3.0381

Numerical solutions of four coupled algebraic equations (V.15)–(V.18) for some sets of coprime integers  $(n_1, n_2, m_1, m_2)$ , under the conditions  $x_3 = y_2 + y_3 - x_1 - x_2$ ,  $x_1 < x_2 < x_3$ ,  $0 < y_2 < y_3$  and  $x_2 < y_2$  for the real roots  $x_i$  and  $y_i$  ( $i = 1, 2, 3$ ).

## 2. Generalization of $Y^{p,q}$

Making use of the five-dimensional minimal gauged supergravity solution (V.1), we have discussed the global metrics on compact manifolds  $M_5$  and it has been seen that they can be regarded as a generalization of  $L^{a,b,c}$ . In the special case, we can also, and this time rather precisely discuss the global properties of the metrics which can be regarded as a generalization of  $Y^{p,q}$  [4]. Taking

a certain limit of the solution (V.1), we obtain the metric locally given by

$$g = (\xi - x)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dx^2}{Q(x)} + Q(x)(d\psi_1 + \cos \theta d\phi)^2 + 4 \left( d\psi_0 + \left( x + \frac{q}{x - \xi} \right) d\psi_1 + \left( x - \xi + \frac{q}{x - \xi} \right) \cos \theta d\phi \right)^2, \quad (\text{V.21})$$

where

$$Q(x) = \frac{4x^3 + (1 - 12\xi)x^2 + (8q - 2\xi + 12\xi^2)x + k}{\xi - x} \quad (\text{V.22})$$

and  $q$ ,  $\xi$  and  $k$  are free parameters. The metric is again a Sasaki with torsion metric and satisfies the equations of motion of five-dimensional minimal gauged supergravity with the Maxwell potential

$$A_{(1)} = -\frac{2\sqrt{3}q}{x - \xi}(d\psi_1 + \cos \theta d\phi). \quad (\text{V.23})$$

The torsion 3-form is given by  $T = *F_{(2)}/\sqrt{3}$ .

Following [4, 5], we study global properties of the metric (V.21). Before starting the analysis, we perform the following coordinate transformation

$$\psi_1 = -\psi + \alpha, \quad \psi_0 = \xi\psi. \quad (\text{V.24})$$

Then the metric is

$$g_5 = (\xi - x)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dx^2}{Q(x)} + \frac{4\xi^2 Q(x)}{F(x)}(d\psi - \cos \theta d\phi)^2 + F(x) \left( d\alpha - f(x)(d\psi - \cos \theta d\phi) \right)^2, \quad (\text{V.25})$$

where

$$F(x) = Q(x) + 4 \left( x + \frac{q}{x - \xi} \right)^2, \quad (\text{V.26})$$

$$f(x) = \frac{Q(x) + 4 \left( x + \frac{q}{x - \xi} \right) \left( x - \xi + \frac{q}{x - \xi} \right)}{F(x)}. \quad (\text{V.27})$$

It should be noticed that when  $q = 0$  and  $\xi = 1/6$ , the metric is the local form of the Sasaki-Einstein metric  $Y^{p,q}$ . In addition, we obtain the homogeneous Sasaki-Einstein metric  $T^{1,1}$  if we take the coordinate transformation  $x = c/6y$  and send  $c \rightarrow 0$ . We also find that when  $k = -4\xi^3 + \xi^2 - 8q\xi$ , the function  $Q(x)$  degenerates to a polynomial of degree 2 and we have  $Q = -4x^2 + (8\xi - 1)x - 4\xi^2 + \xi - 8q$ . Then the metric is the standard  $S^5$  metric when  $q = 0$ . Otherwise,  $Q(x)$  is a rational function and henceforth we will focus on the case.



The metric  $g_5$  is positive definite when there exist three distinct real roots  $x_1, x_2$  and  $x_3$  of the equation  $Q(x) = 0$  such that

$$x_1 < x_2 < x_3, \quad x_2 < \xi, \quad (\text{V.28})$$

and the coordinate  $x$  takes the range  $x_1 \leq x \leq x_2$ . Although we will show later that the 5-dimensional space  $(M_5, g_5)$  is an  $S^1$ -bundle over 4-dimensional space  $B$  given by the metric

$$g_B = (\xi - x)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dx^2}{Q(x)} + \frac{4\xi^2 Q(x)}{F(x)}(d\psi - \cos \theta d\phi)^2, \quad (\text{V.29})$$

we shall see first that  $g_B$  can extend globally on  $S^2$ -bundle over  $S^2$ . Fixing the coordinates  $\theta$  and  $\phi$  in (V.29) and introducing a new coordinate  $r = 2|x - x_i|^{1/2}/|Q'(x_i)|^{1/2}$ , we can evaluate the behavior near  $x = x_i$  of the fiber metric as

$$dr^2 + \left( \frac{\xi(x_i - \xi)Q'(x_i)}{2(x_i(x_i - \xi) + q)} \right)^2 r^2 d\psi^2. \quad (\text{V.30})$$

Hence, avoiding conical singularities at  $x = x_i$  requires both of the condition

$$\frac{\xi(x_i - \xi)Q'(x_i)}{x_i(x_i - \xi) + q} = \pm n \quad (\text{V.31})$$

and the range of  $\psi$  given by  $0 \leq \psi \leq 4\pi/n$  with a constant  $n \neq 0$ . Eq. (V.31) is explicitly written as

$$(12\xi - n_i)x_i^2 + \xi(2 - 24\xi + n_i)x_i + 2\xi(4q - \xi + 6\xi^2) - qn_i = 0, \quad (i = 1, 2), \quad (\text{V.32})$$

where  $n_i$  take  $\pm n$ , respectively. Thus, two of three parameters  $q, k$  and  $\xi$  are fixed by the regular condition (V.32). Since the Chern number is calculated as

$$c_1(B) = \frac{n}{4\pi} \int_{S^2} d(-\cos \theta d\phi) = n, \quad (\text{V.33})$$

the 4-dimensional space  $B$  is a trivial bundle  $S^2 \times S^2$  for even integer  $n$  and a twisted  $S^2$ -bundle for odd integer  $n$ , respectively. For simplicity, we deal with the case  $n_1 = n_2 = n$ . We notice that Eq. (V.32) becomes trivial when  $q = 0$ ,  $\xi = 1/6$  and  $n = 2$ , which reproduces the Sasaki-Einstein metric  $Y^{p,q}$ . In the case  $\xi \neq n/12$  nor  $n/16$ , we obtain more general solutions of Eq. (V.32).

$$q = \frac{(2 - n)\xi(-n + 4\xi + 4n\xi)}{4(n - 16\xi)(n - 12\xi)}, \quad k = \frac{\xi(-n + 4\xi + 8n\xi - 48\xi^2)L(\xi)}{4(n - 16\xi)(n - 12\xi)^3}, \quad (\text{V.34})$$

where

$$\begin{aligned} L(\xi) = & 2n^2 - n^3 + 4(n^3 - n^2 - 6n)\xi \\ & + 16(n^2 + 16n + 4)\xi^2 - 192(7n + 8)\xi^3 + 9216\xi^4. \end{aligned} \quad (\text{V.35})$$

Then the roots of the function  $Q(x)$ ,  $x_1$ ,  $x_2$  and  $x_3$  are given by

$$x_{1,2} = \frac{2\xi + n\xi - 24\xi^2 \pm \sqrt{\frac{(n-2)n\xi(-n+5n\xi+10\xi-48\xi^2)}{n-16\xi}}}{2(n-12\xi)}, \quad (\text{V.36})$$

$$x_3 = \frac{-n+4\xi+8n\xi-48\xi^2}{4(n-12\xi)}, \quad (\text{V.37})$$

where the choice of the sign in (V.36) depends on the sign of  $n-12\xi$ . The reality condition of  $x_1$  and  $x_2$  and the inequalities (V.28) require the following ranges of  $\xi$  for each integer  $n$ :

$$(a) \quad n \geq 4, \quad \xi_1 < \xi < \frac{n}{4(n+1)}, \quad (\text{V.38})$$

$$(b) \quad n = 1, \quad \frac{15-\sqrt{33}}{96} < \xi < \frac{1}{8}, \quad (\text{V.39})$$

$$(c) \quad n \leq -1, \quad \frac{n}{8} < \xi < \xi_1 \quad \text{or} \quad \xi_2 < \xi < \xi_3, \quad (\text{V.40})$$

where the quantities  $\xi_1$ ,  $\xi_2$  and  $\xi_3$  are defined by

$$\xi_1 = \frac{1}{96}(10+5n-\sqrt{100-92n+25n^2}), \quad (\text{V.41})$$

$$\xi_2 = \frac{1}{96}(10+5n+\sqrt{100-92n+25n^2}), \quad (\text{V.42})$$

$$\xi_3 = \frac{1}{48}(5+n+\sqrt{25-14n+n^2}). \quad (\text{V.43})$$

The regular condition for five-dimensional metric  $g_5$  gives rise to further constraint, under which we must choose the period of the fiber direction  $\alpha$  in (V.25) so as to describe a principal  $S^1$ -bundle over  $B$ . Since the connection 1-form is given by

$$\mathcal{A} = f(x)(d\psi - \cos\theta d\phi) \quad (\text{V.44})$$

the periods  $P_i$  ( $i = 1, 2$ ) are calculated as [5],

$$\begin{aligned} P_1 &= \frac{1}{2\pi} \int_{C_1} d\mathcal{A} = \frac{2}{n}(f(x_2) - f(x_1)), \\ P_2 &= \frac{1}{2\pi} \int_{C_2} d\mathcal{A} = 2f(x_2) \end{aligned} \quad (\text{V.45})$$

where  $n$  is the Chern number given by (V.33) and  $C_1$  and  $C_2$  represent the basis for  $H_2(B, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$  [58]. Now we require

$$\frac{f(x_1)}{f(x_2)} = \frac{\ell}{m}, \quad (\text{V.46})$$

where  $\ell, m \in \mathbb{Z}$ . Then,  $\kappa^{-1}d\mathcal{A}/2\pi$  has integral periods if we set  $\kappa = 2hf(x_2)/(mn)$  with  $h = \text{gcd}(\ell - m, nm)$ . Thus we take the range  $0 \leq \alpha \leq 2\pi\kappa$ . A numerical calculation shows that

our solution (V.34) admits the parameter  $\xi$  satisfying the condition (V.46), and hence the 5-dimensional space  $M_5$  becomes an  $S^1$ -bundle over  $B$  parameterized by three integers  $\ell, m$  and  $n$ . It is straightforward to verify that the following four Killing vectors

$$\begin{aligned} v_1 &= \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi}, \quad v_2 = -\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi}, \\ \ell_1 &= \frac{2}{n} \left( \frac{\partial}{\partial \psi} + f(x_1) \frac{\partial}{\partial \alpha} \right), \quad \ell_2 = \frac{2}{n} \left( \frac{\partial}{\partial \psi} + f(x_2) \frac{\partial}{\partial \alpha} \right), \end{aligned} \quad (\text{V.47})$$

vanish with the surface gravity 1 on the sub-manifolds given by  $\theta = 0, \theta = \pi, x = x_1$  and  $x = x_2$ , respectively, and they have a linear relation

$$(N_1 - N_2)(v_1 + v_2) + nN_2\ell_1 - nN_1\ell_2 = 0 \quad (\text{V.48})$$

with  $N_1 = n\ell/h \in \mathbb{Z}$ ,  $N_2 = nm/h \in \mathbb{Z}$  ( cf. (V.12)).

The volume is given by

$$\text{Vol}(M_5) = \pi^3 \left| \frac{32\xi\kappa(x_2 - x_1)(2\xi - x_1 - x_2)}{n} \right|. \quad (\text{V.49})$$

Moreover, since  $B$  is a simply-connected manifold, it follows that  $M_5$  is also simply-connected. Note also that  $M_5$  is a spin manifold [5]. Smale's theorem states that any simply-connected compact five-manifold which is spin and has no torsion in the second homology group is diffeomorphic to  $S^5 \# k(S^2 \times S^3)$  for some non-negative integer  $k$ . Thus, together with the analysis in App. A of [4], we see that  $M_5$  is topologically  $S^2 \times S^3$ .

## B. Non-compact manifolds in eleven dimensions

Next, we turn to discussing the global structure of the eleven-dimensional supergravity solution. We assume that the functions  $X_\mu(x_\mu)$  take the form

$$\begin{aligned} X(x) &\equiv X_1(x_1) = c(x - a)P(x), \\ Y_k(y_k) &\equiv X_{k+1}(x_{k+1}) = c \prod_{i=1}^5 (y_k - \beta_i), \quad k = 1, \dots, 4, \end{aligned} \quad (\text{V.50})$$

where  $P(x)$  is a positive definite polynomial of degree 4 and  $a, c, \beta_i$  are real constants satisfying

$$c > 0, \quad \beta_1 < \beta_2 < \dots < \beta_5 < a. \quad (\text{V.51})$$

Then we choose the region of the coordinates  $x, y_k$  as

$$\beta_1 \leq y_1 \leq \beta_2 \leq \dots \leq y_4 \leq \beta_5 < a \leq x < \infty. \quad (\text{V.52})$$

The fact that the region of  $x$  is infinite corresponds to non-compactness of manifold. Then the metric is positive definite except for the boundaries  $y_k = \beta_k, y_k = \beta_{k+1}$  and  $x = a$ . From App. B we see that the curvature is finite at the points  $y_k = y_{k+1} = \beta_k$ . Some calculations analogous to the five-dimensional case yield that the following vector fields are Killing vector fields vanishing at the boundaries  $x = a, y_k = \beta_k$  and  $y_k = \beta_{k+1}$  ( $k=1,2,3,4$ ), respectively,

$$\begin{aligned} v_0 &= \frac{2}{X'(a)} \left( (N_1(a) + a^5) \frac{\partial}{\partial \psi_0} + \sum_{\ell=1}^5 (-1)^k \ell a^{5-\ell} \frac{\partial}{\partial \psi_\ell} \right), \\ v_k &= \frac{2}{Y'_k(\beta_k)} \left( (N_{k+1}(\beta_k) + \beta_k^5) \frac{\partial}{\partial \psi_0} + \sum_{\ell=1}^5 (-1)^\ell \beta_k^{5-\ell} \frac{\partial}{\partial \psi_\ell} \right), \\ w_k &= \frac{2}{Y'_k(\beta_{k+1})} \left( (N_{k+1}(\beta_{k+1}) + \beta_{k+1}^5) \frac{\partial}{\partial \psi_0} + \sum_{\ell=1}^5 (-1)^\ell \beta_{k+1}^{5-\ell} \frac{\partial}{\partial \psi_\ell} \right). \end{aligned} \quad (\text{V.53})$$

These Killing vector fields have a unit surface gravity. If we impose the condition  $q_2 = q_3 = q_4 = q_5$ , then we have  $N_2 = N_3 = N_4 = N_5$ , which implies the relation  $v_k = w_{k-1}$  ( $k = 2, 3, 4$ ). Hence we can use them as the new Killing coordinates  $\phi_\alpha$  with period  $2\pi$  representing the canonical coordinate of torus  $T^6$ ,

$$\frac{\partial}{\partial \phi_0} = v_0, \quad \frac{\partial}{\partial \phi_1} = v_1, \quad \frac{\partial}{\partial \phi_k} = v_k = w_{k-1} \quad (k = 2, 3, 4), \quad \frac{\partial}{\partial \phi_5} = \omega_4. \quad (\text{V.54})$$

## VI. SUMMARY AND DISCUSSIONS

We have discussed a deformation of Sasakian structure in the presence of totally skew-symmetric torsion by introducing odd dimensional manifolds whose metric cones are Kähler with torsion (KT) manifolds. We call such manifolds Sasaki with torsion (ST) manifolds. The ST manifolds admit a normal almost contact metric structure whose Reeb vector  $\xi$  is a Killing vector field of unit length.

We have presented an example of the  $(2n + 1)$ -dimensional ST metric. The metric is quasi Sasakian and admits  $n + 1$  Killing vector fields preserving the KT structure. We also have demonstrated that there exist two kinds of hidden symmetries; one is given by special Killing forms and other is given by GKY tensors which are related to non-trivial rank-2 Killing tensors. Although the former exists in the general ST manifold, the existence of the GKY tensors could not be always expected. Indeed, our metric is the first example admitting non-trivial odd-rank GCKY tensors. It would be interesting to examine in this geometry whether the GKY tensors generate symmetry operators for the Klein-Gordon and Dirac operators.

Using the metric (III.1) as an *ansatz*, we have constructed exact solutions in five-dimensional minimal (un-)gauged supergravity and eleven-dimensional supergravity, and discussed the global

structures of the solutions. In particular the ST metrics on the five-dimensional compact manifolds provide a natural generalization of the toric Sasaki-Einstein metrics  $Y^{p,q}$  and  $L^{a,b,c}$ . For these metrics there exists a  $T^3$  action preserving the KT structure and the Einstein condition is replaced by the equations of motion of the minimal gauged supergravity.

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### Appendix A: Some properties of T-contact metric manifolds

We show some useful formulae in order to prove Propositions in Sec. II. Let us remind that  $(M, g, T, \xi, \eta, \Phi)$  is an almost contact metric manifold equipped with a 3-form  $T$  satisfying (II.13) and define a fundamental 2-form  $\omega$  by  $\omega(X, Y) = g(\Phi(X), Y)$ . Then a straightforward calculation leads us to

$$\begin{aligned} 2g((\nabla_X^T \Phi)Y, Z) = & -d\omega(X, \Phi Y, \Phi Z) + d\omega(X, Y, Z) + M(X, Y, Z) \\ & + g(N^{(1)}(Y, Z), \Phi X) + \eta(X)N^{(2)}(Y, Z) \\ & + d^T \eta(X, \Phi Z)\eta(Y) - d^T \eta(X, \Phi Y)\eta(Z) , \end{aligned} \quad (\text{A.1})$$

where we have introduced tensor fields  $N^{(i)}$  ( $i = 1, 2$ ) defined in Sec. 6 of [32]:

$$N^{(1)}(X, Y) = N_\Phi(X, Y) + d\eta(X, Y)\xi , \quad (\text{A.2})$$

$$N^{(2)}(X, Y) = (\mathcal{L}_{\Phi X}\eta)(Y) - (\mathcal{L}_{\Phi Y}\eta)(X) , \quad (\text{A.3})$$

and defined a tensor  $M$  by

$$M(X, Y, Z) = T(X, \Phi Y, Z) + T(X, Y, \Phi Z) - T(\xi, X, \Phi Y)\eta(Z) + T(\xi, X, \Phi Z)\eta(Y) . \quad (\text{A.4})$$

Note that

$$M(\xi, X, Y) = M(X, \xi, Y) = M(X, Y, \xi) = 0 . \quad (\text{A.5})$$

In the case of a T-contact metric manifold, this formula simplifies. Indeed, we have

$$\begin{aligned} N^{(2)}(X, Y) &= d\eta(X, \Phi Y) + d\eta(\Phi X, Y) \\ &= d^T \eta(X, \Phi Y) + d\eta^T(\Phi X, Y) + T(\xi, X, \Phi Y) + T(\xi, \Phi X, Y) = 0 , \end{aligned} \quad (\text{A.6})$$

where we have used (II.13), (II.25) and (II.27) at the last equality.

Furthermore, we have

$$\xi \lrcorner d^T \eta = \xi \lrcorner d\eta = 0 , \quad (\text{A.7})$$

which implies  $\mathcal{L}_\xi \eta = 0$  and  $\mathcal{L}_\xi d\eta = 0$ . We also see

$$\mathcal{L}_\xi d^T \eta = d\xi \lrcorner d^T \eta + \xi \lrcorner dd^T = 2\xi \lrcorner d\omega = 0 , \quad (\text{A.8})$$

which leads to

$$2(\mathcal{L}_\xi g)(X, Y) = d^T \eta(X, (\mathcal{L}_\xi \Phi)(Y)) . \quad (\text{A.9})$$

Thus  $\xi$  is a Killing vector field if and only if  $\mathcal{L}_\xi \Phi = 0$ .

## Appendix B: Riemann, Ricci and scalar curvatures

In this section we collect some technical results.

### 1. The Levi-Civita connection

For the orthonormal frame (III.9) of the metric (III.1), we obtain the connection 1-forms (III.10)–(III.15). By using relation  $\nabla_{e_a} e^b(e_c) = -\omega^b_c(e_a)$ , we can compute the covariant derivatives

as follows:

$$\nabla_{e_\mu} e_\mu = \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_\rho, \quad (\text{B.1})$$

$$\nabla_{e_\mu} e_\nu = -\frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e_\mu, \quad \mu \neq \nu \quad (\text{B.2})$$

$$\nabla_{e_\mu} e_{\hat{\mu}} = \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_{\hat{\rho}} - (1 + \partial_\mu H) e_0, \quad (\text{B.3})$$

$$\nabla_{e_\mu} e_{\hat{\nu}} = -\frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e_{\hat{\mu}}, \quad \mu \neq \nu \quad (\text{B.4})$$

$$\nabla_{e_{\hat{\mu}}} e_\mu = \partial_\mu \sqrt{Q_\mu} e_{\hat{\mu}} - \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_{\hat{\rho}} + (1 + \partial_\mu H) e_0, \quad (\text{B.5})$$

$$\nabla_{e_{\hat{\mu}}} e_\nu = -\frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e_{\hat{\mu}} + \frac{\sqrt{Q_\mu}}{2(x_\mu - x_\nu)} e_{\hat{\nu}}, \quad \mu \neq \nu \quad (\text{B.6})$$

$$\nabla_{e_{\hat{\mu}}} e_{\hat{\mu}} = -\partial_\mu \sqrt{Q_\mu} e_\mu + \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_\rho, \quad (\text{B.7})$$

$$\nabla_{e_{\hat{\mu}}} e_{\hat{\nu}} = \frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e_\mu - \frac{\sqrt{Q_\mu}}{2(x_\mu - x_\nu)} e_\nu, \quad \mu \neq \nu \quad (\text{B.8})$$

$$\nabla_{e_\mu} e_0 = (1 + \partial_\mu H) e_{\hat{\mu}}, \quad (\text{B.9})$$

$$\nabla_{e_{\hat{\mu}}} e_0 = -(1 + \partial_\mu H) e_\mu, \quad (\text{B.10})$$

$$\nabla_{e_0} e_\mu = (1 + \partial_\mu H) e_{\hat{\mu}}, \quad (\text{B.11})$$

$$\nabla_{e_0} e_{\hat{\mu}} = -(1 + \partial_\mu H) e_\mu, \quad (\text{B.12})$$

$$\nabla_{e_0} e_0 = 0, \quad (\text{B.13})$$

where the function  $H$  is given by (III.16).

Since the curvature 2-form  $\mathcal{R}^a_b$  is defined by the second structure equation

$$\mathcal{R}^a_b = d\omega^a_b + \sum_c \omega^a_c \wedge \omega^c_b, \quad (\text{B.14})$$

we obtain

$$\begin{aligned} \mathcal{R}^\mu_\nu = & K_{\mu\nu} e^\mu \wedge e^\nu + \left( K_{\mu\nu} - (1 + \partial_\mu H)(1 + \partial_\nu H) \right) e^{\hat{\mu}} \wedge e^{\hat{\nu}} \\ & - \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\nu} e^{\hat{\mu}} \wedge e^0 + \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\mu} e^{\hat{\nu}} \wedge e^0, \quad (\mu \neq \nu) \end{aligned} \quad (\text{B.15})$$

$$\begin{aligned} \mathcal{R}^\mu_{\hat{\mu}} = & -\frac{1}{2} \left( \partial_\mu^2 Q_T + 6(1 + \partial_\mu H)^2 \right) e^\mu \wedge e^{\hat{\mu}} + 2 \sum_{\nu \neq \mu} \left( K_{\mu\nu} - (1 + \partial_\mu H)(1 + \partial_\nu H) \right) e^\nu \wedge e^{\hat{\nu}} \\ & - \sqrt{Q_\mu} \partial_\mu^2 H e^\mu \wedge e^0 - \sum_{\nu \neq \mu} \frac{\partial_\mu H - \partial_\nu H}{x_\mu - x_\nu} \sqrt{Q_\nu} e^\nu \wedge e^0 \end{aligned} \quad (\text{B.16})$$

$$\begin{aligned}\mathcal{R}^\mu_{\hat{\nu}} = & K_{\mu\nu} e^\mu \wedge e^{\hat{\nu}} + \left( K_{\mu\nu} - (1 + \partial_\mu H)(1 + \partial_\nu H) \right) e^\nu \wedge e^{\hat{\mu}} \\ & - \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\nu} e^\mu \wedge e^0 - \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\mu} e^\nu \wedge e^0, \quad (\mu \neq \nu)\end{aligned}\quad (\text{B.17})$$

$$\begin{aligned}\mathcal{R}^{\hat{\mu}}_{\hat{\nu}} = & K_{\mu\nu} e^{\hat{\mu}} \wedge e^{\hat{\nu}} + \left( K_{\mu\nu} - (1 + \partial_\mu H)(1 + \partial_\nu H) \right) e^\mu \wedge e^\nu \\ & - \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\nu} e^{\hat{\mu}} \wedge e^0 + \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\mu} e^{\hat{\nu}} \wedge e^0, \quad (\mu \neq \nu)\end{aligned}\quad (\text{B.18})$$

$$\begin{aligned}\mathcal{R}^\mu_0 = & -\sqrt{Q_\mu} \partial_\mu^2 H e^\mu \wedge e^{\hat{\mu}} - \sum_{\nu \neq \mu} \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\nu} e^\nu \wedge e^{\hat{\mu}} \\ & - \sum_{\nu \neq \mu} \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\nu} e^\mu \wedge e^{\hat{\nu}} - \sum_{\nu \neq \mu} \frac{\partial_\mu H - \partial_\nu H}{x_\mu - x_\nu} \sqrt{Q_\mu} e^\nu \wedge e^{\hat{\nu}} \\ & + (1 + \partial_\mu H)^2 e^\mu \wedge e^0\end{aligned}\quad (\text{B.19})$$

$$\begin{aligned}\mathcal{R}^{\hat{\mu}}_0 = & -\sum_{\nu \neq \mu} \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\nu} e^\mu \wedge e^\nu - \sum_{\nu \neq \mu} \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\mu} e^{\hat{\mu}} \wedge e^{\hat{\nu}} \\ & + (1 + \partial_\mu H)^2 e^{\hat{\mu}} \wedge e^0,\end{aligned}\quad (\text{B.20})$$

where

$$K_{\mu\nu} \equiv \frac{-\partial_\mu Q_T + \partial_\nu Q_T}{4(x_\mu - x_\nu)}, \quad Q_T \equiv \sum_{\mu=1}^n Q_\mu. \quad (\text{B.21})$$

The Ricci curvature is defined by

$$Ric(e_a, e_b) = \sum_c \mathcal{R}^c_a(e_c, e_b). \quad (\text{B.22})$$

Thus nonzero components of the Ricci curvature are

$$Ric(e_\mu, e_\mu) = -\frac{1}{2} \partial_\mu^2 Q_T + 2 \sum_{\nu \neq \mu} K_{\mu\nu} - 2(1 + \partial_\mu H)^2, \quad (\text{B.23})$$

$$Ric(e_{\hat{\mu}}, e_{\hat{\mu}}) = -\frac{1}{2} \partial_\mu^2 Q_T + 2 \sum_{\nu \neq \mu} K_{\mu\nu} - 2(1 + \partial_\mu H)^2, \quad (\text{B.24})$$

$$Ric(e_0, e_0) = 2 \sum_{\mu=1}^n (1 + \partial_\mu H)^2. \quad (\text{B.25})$$

Note that the Ricci curvature is diagonalized. The scalar curvature is defined by

$$scal = \sum_a Ric(e_a, e_a). \quad (\text{B.26})$$

Thus we obtain

$$scal = -\sum_{\mu=1}^n \partial_\mu^2 Q_T + 4 \sum_{\mu \neq \nu} K_{\mu\nu} - 2 \sum_{\mu=1}^n (1 + \partial_\mu H)^2. \quad (\text{B.27})$$



## 2. The connection with totally skew-symmetric torsion

Next, we compute the curvature quantities with torsion with respect to the orthonormal frame (III.9) of the metric (III.1). Since we have obtained the covariant derivatives with respect to the Levi-Civita connection  $\nabla$ , (B.1)–(B.13), hence we can compute from Eq. (II.1) the covariant derivatives with respect to the torsion connection  $\nabla^T$  as

$$\nabla_{e_a}^T e_b = \nabla_{e_a} e_b + \frac{1}{2} T(e_a, e_b) . \quad (\text{B.28})$$

Thus we obtain

$$\nabla_{e_\mu}^T e_\mu = \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_\rho , \quad (\text{B.29})$$

$$\nabla_{e_\mu}^T e_\nu = -\frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e_\mu , \quad (\text{B.30})$$

$$\nabla_{e_\mu}^T e_{\hat{\mu}} = \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_{\hat{\rho}} - e_0 , \quad (\text{B.31})$$

$$\nabla_{e_\mu}^T e_{\hat{\nu}} = -\frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e_{\hat{\mu}} , \quad (\text{B.32})$$

$$\nabla_{e_{\hat{\mu}}}^T e_\mu = \partial_\mu \sqrt{Q_\mu} e_{\hat{\mu}} - \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_{\hat{\rho}} + e_0 , \quad (\text{B.33})$$

$$\nabla_{e_{\hat{\mu}}}^T e_\nu = -\frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e_{\hat{\mu}} + \frac{\sqrt{Q_\mu}}{2(x_\mu - x_\nu)} e_{\hat{\nu}} , \quad (\text{B.34})$$

$$\nabla_{e_{\hat{\mu}}}^T e_{\hat{\mu}} = -\partial_\mu \sqrt{Q_\mu} e_\mu + \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_\rho , \quad (\text{B.35})$$

$$\nabla_{e_{\hat{\mu}}}^T e_{\hat{\nu}} = \frac{\sqrt{Q_\nu}}{2(x_\mu - x_\nu)} e_\mu - \frac{\sqrt{Q_\mu}}{2(x_\mu - x_\nu)} e_\nu , \quad (\text{B.36})$$

$$\nabla_{e_\mu}^T e_0 = e_{\hat{\mu}} , \quad (\text{B.37})$$

$$\nabla_{e_{\hat{\mu}}}^T e_0 = -e_\mu , \quad (\text{B.38})$$

$$\nabla_{e_0}^T e_\mu = (1 + 2\partial_\mu H) e_{\hat{\mu}} , \quad (\text{B.39})$$

$$\nabla_{e_0}^T e_{\hat{\mu}} = -(1 + 2\partial_\mu H) e_\mu , \quad (\text{B.40})$$

$$\nabla_{e_0}^T e_0 = 0 , \quad (\text{B.41})$$

where the function  $H$  is again given by (III.16).

### Appendix C: Calabi-Yau with torsion metric on a cone

We begin with the metric (III.23) and choose the same orthonormal frame as (III.24), then the connection 1-forms are calculated as (III.25). For the Hermitian connection  $\bar{\nabla}^B$  with respect to the Bismut torsion (III.29), the connection 1-form with torsion  $\bar{\omega}^{B\alpha}{}_{\beta}$  are calculated as

$$\bar{\omega}^{B\alpha}{}_{\beta} = \bar{\omega}^{\alpha}{}_{\beta} - \frac{1}{2} \sum_{\gamma} B^{\alpha}{}_{\beta\gamma} \bar{e}^{\gamma} . \quad (\text{C.1})$$

That is, we have

$$\bar{\omega}^{Br}{}_a = -\frac{\bar{e}^a}{r} , \quad (\text{C.2})$$

$$\bar{\omega}^{B\mu}{}_{\nu} = -\frac{\sqrt{Q_{\nu}}}{2(x_{\mu} - x_{\nu})} \frac{\bar{e}^{\mu}}{r} - \frac{\sqrt{Q_{\mu}}}{2(x_{\mu} - x_{\nu})} \frac{\bar{e}^{\nu}}{r} , \quad \mu \neq \nu \quad (\text{C.3})$$

$$\bar{\omega}^{B\mu}{}_{\hat{\mu}} = -\partial_{\mu} \sqrt{Q_{\mu}} \frac{\bar{e}^{\hat{\mu}}}{r} + \sum_{\nu \neq \mu} \frac{\sqrt{Q_{\nu}}}{2(x_{\mu} - x_{\nu})} \frac{\bar{e}^{\hat{\nu}}}{r} - (1 + 2\partial_{\mu} H) \frac{\bar{e}^0}{r} , \quad (\text{C.4})$$

$$\bar{\omega}^{B\mu}{}_{\hat{\nu}} = \frac{\sqrt{Q_{\nu}}}{2(x_{\mu} - x_{\nu})} \frac{\bar{e}^{\hat{\mu}}}{r} - \frac{\sqrt{Q_{\mu}}}{2(x_{\mu} - x_{\nu})} \frac{\bar{e}^{\hat{\nu}}}{r} , \quad \mu \neq \nu \quad (\text{C.5})$$

$$\bar{\omega}^{B\hat{\mu}}{}_{\hat{\nu}} = -\frac{\sqrt{Q_{\nu}}}{2(x_{\mu} - x_{\nu})} \frac{\bar{e}^{\mu}}{r} - \frac{\sqrt{Q_{\mu}}}{2(x_{\mu} - x_{\nu})} \frac{\bar{e}^{\nu}}{r} , \quad \mu \neq \nu \quad (\text{C.6})$$

$$\bar{\omega}^{B\mu}{}_0 = -\frac{\bar{e}^{\hat{\mu}}}{r} , \quad (\text{C.7})$$

$$\bar{\omega}^{B\hat{\mu}}{}_0 = \frac{\bar{e}^{\mu}}{r} . \quad (\text{C.8})$$

Note that if we restrict the connection 1-forms  $\bar{\omega}^{B\alpha}{}_{\beta}$  on the hyperplane of  $r = 1$ , then we obtain the connection 1-form  $\omega^{Ta}{}_b = \bar{\omega}^{Ba}{}_b|_{r=1}$  with respect to the original metric  $g_{2n+1}$  and the torsion  $T$ . Since the curvature 2-form with torsion  $\bar{\mathcal{R}}^{B\alpha}{}_{\beta}$  and the Ricci form with torsion  $\rho^B(X, Y)$  are calculated as [55]

$$\bar{\mathcal{R}}^{B\alpha}{}_{\beta}(X, Y) = g(\bar{R}^B(X, Y) \bar{e}_{\alpha}, \bar{e}_{\beta}) , \quad (\text{C.9})$$

$$\rho^B(X, Y) = \frac{1}{2} \sum_{\alpha} \bar{\mathcal{R}}^B(X, Y, \bar{e}_{\alpha}, J(\bar{e}_{\alpha})) , \quad (\text{C.10})$$

where  $\bar{R}^B(X, Y)$  is the curvature defined by (II.17) with respect to  $\bar{\nabla}^B$ , we have the curvature 2-form with torsion as

$$\bar{\mathcal{R}}^{Br}{}_0 = -2 \sum_{\mu=1}^n \partial_{\mu} H e^{\mu} \wedge e^{\hat{\mu}} , \quad (\text{C.11})$$

$$\begin{aligned} \bar{\mathcal{R}}^{B\mu}{}_{\hat{\mu}} &= -\frac{1}{2} \left( \partial_{\mu}^2 Q_T + 4 \right) e^{\mu} \wedge e^{\hat{\mu}} + \frac{1}{2} \sum_{\nu \neq \mu} \left( -\frac{\partial_{\mu} Q_T}{x_{\mu} - x_{\nu}} + \frac{\partial_{\nu} Q_T}{x_{\mu} - x_{\nu}} \right) e^{\nu} \wedge e^{\hat{\nu}} \\ &\quad - 2 \sum_{\nu=1}^n \sqrt{Q_{\nu}} \partial_{\mu} \partial_{\nu} H e^{\nu} \wedge e^0 - 2 \sum_{\nu=1}^n (1 + 2\partial_{\mu} H)(1 + \partial_{\nu} H) e^{\nu} \wedge e^{\hat{\nu}} , \end{aligned} \quad (\text{C.12})$$

and the non-zero components of the Ricci form as

$$\rho^B(e_\mu, e_{\bar{\mu}}) = -\frac{1}{2}\partial_\mu^2 Q_T + \frac{1}{2}\sum_{\nu \neq \mu} \left( -\frac{\partial_\mu Q_T}{x_\mu - x_\nu} + \frac{\partial_\nu Q_T}{x_\mu - x_\nu} \right) - 2(n+1)(1 + \partial_\mu H) . \quad (\text{C.13})$$

Thus we find that  $\rho^B(X, Y) = 0$  for all vector fields  $X, Y$ , when provided that the functions  $X_\mu$  and  $N_\mu$  take the form

$$X_\mu(x_\mu) = -4x_\mu^{n+1} + \sum_{j=1}^n c_j x_\mu^j + b_\mu - 4(n+1)q_\mu x_\mu , \quad (\text{C.14})$$

$$N_\mu(x_\mu) = \sum_{i=1}^{n-1} a_i x_\mu^i + q_\mu , \quad (\text{C.15})$$

where  $a_i, b_j, m_\mu$  and  $q_\mu$  are constant parameters. This gives a Calabi-Yau with torsion metric on a cone. The function (C.14) in five dimensions is different from (IV.10).

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